

A p -adic Approach to the Weil Representation of Discriminant Forms Arising from Even Lattices

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Introduction

Let M be an even lattice, with dual M^* and level N . The group $Mp_2(\mathbb{Z})$, a double cover of $SL_2(\mathbb{Z})$, admits a representation ρ_M , called the *Weil Representation*, on the space $\mathbb{C}[M^*/M]$. This representation arises naturally in the theory of Siegel theta functions, since a neat description of the theta function of the lattice M is given by a $\mathbb{C}[M^*/M]$ -valued function. This theta function is a modular form with representation ρ_M (see, for example, Theorem 4.1 of [B1]). Since these theta functions have various applications in mathematics (for example, important Number-Theoretic applications can be found in [B2] and [Z2]), the Weil representation appears in many works in various branches of mathematics.

Several properties of the Weil representation have been known for a long time. For example, the fact the Weil representation factors through the finite quotient $Mp_2(\mathbb{Z}/N\mathbb{Z})$ is already given, in a different presentation, in [Scho]. Moreover, the seminal paper [W], which initiated the much more general theory of Weil representations, provides formulae for the representation of matrices in which $c = 0$ or in which c is invertible (see Eq. (16) of that reference). Finally, two recent papers give the formulae for the action of a general element of $Mp_2(\mathbb{Z})$ via ρ_M : See [Sche] for the even signature case and [Str] for the general case. We also note that Proposition 1.6 of [Sh] provides a formula for the action of a general element via ρ_M using a Gauss sum which is not explicitly evaluated.

In all these works theta functions play an essential role. Indeed, they are used to prove the factoring claim in [Scho] as well as in the more general work in [Kl]. [B3] also uses theta functions to prove assertions about Weil representations. Later, the factoring property is used in [Sche] and in [Str] to prove their formulae. The action of elements of the form ST^mST^n is explicitly calculated there, and then one delicately follows the roots of unity in order to evaluate the action of a

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general element of $SL_2(\mathbb{Z})$ or $Mp_2(\mathbb{Z})$. The formula in [Sh] is also proved using theta functions.

In this paper we take a different approach to the evaluation of the ρ_M -action of a general element of $Mp_2(\mathbb{Z})$. This approach was proposed to the author by E. Lapid. First we decompose the Weil representation ρ_M into p -parts, and note that each p -part can be seen as subspace of the Schwartz functions on the p -adic vector space $M_{\mathbb{Q}_p}$ which is preserved under the action of the Weil S^1 -cover of $SL_2(\mathbb{Z}_p)$. Then we demonstrate the power of this method in obtaining a short, neat proof of the factoring claim. This proof uses only basic topological group theory, without referring any theta functions. The final formulae are obtained by evaluating each of the p -parts, considered as the action of matrices with p -adic entries. In this case the integral appearing in Eq. (16) of [W] is reduced to a Gauss sum, which can be explicitly evaluated.

An important ingredient of such a method is knowing precisely which elements of S^1 -cover of $SL_2(\mathbb{F})$, for \mathbb{F} a local field of characteristic different from 2, lie in the metaplectic double cover acting on an \mathbb{F} -vector space with a non-degenerate quadratic form. This questing is interesting on its own right, and in a way it has been answered (in greater generality) using symplectic notation in [Ra] (especially Section 5 there). However, having the additional structure of the quadratic form on the vector space allows one to obtain neater formulae, using the Weil index of the quadratic form and of associated quadratic forms, directly from the theory of [W]. Moreover, it seems simpler to obtain the results in this manner than translating the formulae of [Ra] to this context, at least for $SL_2(\mathbb{F})$. We then show, more or less following [Ku2] and [Ge], that the metaplectic cover splits over the ring of integers wherever the residue field has characteristic different from 2. In fact, we give a simpler proof of this result. We then combine all these ingredients in order to deduce the formulae of [Sche] and [Str]. Along the way we prove also the formula of [Sh]. The final formula is essentially obtained by evaluating explicitly the Gauss sum appearing in [Sh].

We note three important points. First, both in [Sche] and in [Str] the general root of unity appears as the product of “ p -adic factors”. Hence apparently one cannot avoid a (maybe implicit) p -adic decomposition. Second, [Sche] and [Str] work with general finite quadratic modules, while we assume here that an even lattice is given. This does not restrict the generality, since it is well-known that any finite quadratic module is the discriminant form of some even lattice (see [N], for example). Third, observe that our p -adic factors do *not* coincide with those of [Sche] and [Str]. However, we indicate in the end why their *total product* does give the same result as in [Sche] and [Str].

The paper is divided into 9 sections. In Section 1 we go over the basic definitions of lattices and the corresponding Weil representation. In Section 2 we present the decomposition into p -parts, and the identification with a subspace of $\mathcal{S}(M_{\mathbb{Q}_p})$. In Section 3 we prove the factoring claim. In Section 4 we evaluate generalized quadratic Gauss sums, describe Jordan decompositions of p -adic lattices, and relate some Gauss sums arising from them to certain Weil indices. In Section 5 we survey the Weil representation over a local field $\mathbb{F} \neq \mathbb{C}$, and give

the explicit formulae for the metaplectic double cover $Mp_2(\mathbb{F})$ of $SL_2(\mathbb{F})$. In Section 6 we lift (a certain congruence subgroup of) SL_2 over the ring of integers in a non-archimedean local field \mathbb{F} into $Mp_2(\mathbb{F})$, and show that the non-trivial double covers of $SL_2(\mathbb{Z})$ in $Mp_2(\mathbb{R})$ and in $Mp_2(\mathbb{Q}_2)$ are isomorphic. In Section 7 we evaluate the operators from the theory of [W] on certain Schwartz functions on vector spaces over non-archimedean local fields. In Section 8 we obtain our main results and indicate how they correspond to those of [Sche] and [Str] (and in some cases to [B3] as well). Finally, in Section 9 we show what theory we have for odd lattices, and discuss some further possible generalizations.

I am deeply indebted to E. Lapid for his proposal to look for a p -adic proof to the factoring claim, which initiated my work on this paper (and the corresponding part in my Ph.D. thesis). I would also like to thank my Ph.D. advisor R. Livné and to H. M. Farkas for their help. I also thank J. Bruinier, N. Scheithauer and F. Strömberg for fruitful discussions while writing this paper and for referring me to [Ku1]. Special thanks are to T. Yang, for referring me to [Ra].

1 Even Lattices and Weil Representations

In this section we give the basic definitions of lattices, the real and integral metaplectic groups, and the Weil representation.

1.1 Lattices

Throughout this paper, for a commutative ring R with unit, a R -lattice will be a finite rank free R -module M endowed with a symmetric non-degenerate bilinear form, denoted $(\cdot, \cdot) : M \times M \rightarrow R$. The rank of M will be denoted $rk(M)$. We shall also use the shorthand λ^2 to denote (λ, λ) for $\lambda \in M$. For any ring S containing R , we shorthand $M \otimes_R S$ to simply M_S , with the extended bilinear form again denoted $(\cdot, \cdot) : M_S \times M_S \rightarrow S$. A *lattice* is just a \mathbb{Z} -lattice, and a p -adic lattice is a \mathbb{Z}_p -lattice. If R is the ring of integers in a global field \mathbb{K} and v is a non-archimedean place of \mathbb{K} with ring of integers \mathcal{O}_v then for an R -lattice M we shorthand the tensor product $M_{\mathcal{O}_v}$ even further and write simply M_v . The definition we made is perhaps not the most general one, but we shall be interested in this paper only in the cases where R is a principal ideal domain, i.e., \mathbb{Z} , a field, or the ring of integers in a non-archimedean local field, this definition suffices. for our purposes.

For any R -lattice M we denote its *dual* $Hom(M, R)$ by M^* . If R is an integral domain with field of fractions \mathbb{K} then M and M^* can be identified as submodules of the \mathbb{K} -lattice (\mathbb{K} -vector space) $M_{\mathbb{K}}$, such that M^* contains M . Then $D_M = M^*/M$ is a torsion R -module of finite rank, and the bilinear form on M gives a symmetric, non-degenerate \mathbb{K}/R -bilinear on D_M . If R is the ring of integers in a local or global field (e.g., \mathbb{Z} or \mathbb{Z}_p) then D_M is finite, and we denote its cardinality by Δ_M .

If 2 is not a zero-divisor in R then we call an R -lattice M *even* if λ^2 is even (i.e., divisible by 2 in R) for every $\lambda \in M$. In particular, if $2 \in R^*$ (the group of invertible elements in R) then every R -lattice is even. Clearly, if S contains R and 2 is not a zero-divisor in S as well then M_S is an even S -lattice. For any even lattice M the map $q : \lambda \mapsto \frac{\lambda^2}{2}$ is a well-defined quadratic form on M , and if R is an integral domain then it can be extended to M^* , giving a \mathbb{K}/R -valued quadratic form on D_M . For $R = \mathbb{Z}$ this makes D_M a finite quadratic module in the language of [Str] and a discriminant form in the language of [Sche]. We note that if R is the ring of integers in a global field \mathbb{K} then an R -lattice M is even if and only if M_v is even for every place v of \mathbb{K} , i.e., if and only if M_v is even for any place v lying over 2. A lattice which is not even will be called *odd*.

The *level* N of an R -lattice M is defined to be the ideal consisting of all the elements a of R such that $a\frac{\gamma^2}{2} \in R$ for any $\gamma \in M^*$. It follows that $a(\gamma, \delta) \in R$ for any γ and δ in M^* . It is clear that if R is the ring of integers in a global field \mathbb{K} then the level of M_v is $N \otimes_R \mathcal{O}_v$. For $R = \mathbb{Z}$ or $R = \mathbb{Z}_p$ we use the slight abuse of notation in which N may denote either the level as an ideal or a generator of it. Clearly, an even lattice is *unimodular* if and only if it has level 1 (see more generally Lemma 2.1 below), but this statement is false for odd lattices.

For a \mathbb{Z} -lattice M we define its *signature* $\text{sgn}(M)$ to be the signature of $M_{\mathbb{R}}$. Its image modulo 8 is what is referred to as the signature of D_M in [Sche] and [Str]. Adopting the notation $\mathbf{e}(z) = e^{2\pi iz}$ for complex z and denoting the root of unity $\mathbf{e}(\frac{1}{8})$ (which will appear many times in this paper) by ζ_8 , we can now quote *Milgram's formula*, which evaluates a certain Gauss sum corresponding to the even lattice M (or to its discriminant form, depending on the point of view) and states that

$$\sum_{\gamma \in D_M} \mathbf{e}\left(\frac{\gamma^2}{2}\right) = \zeta_8^{\text{sgn}(M)} \sqrt{\Delta_M}.$$

1.2 The Metaplectic Groups $Mp_2(\mathbb{R})$ and $Mp_2(\mathbb{Z})$

The group $SL_2(\mathbb{R})$ admits a non-trivial double cover $Mp_2(\mathbb{R})$, which has several equivalent descriptions. The fact that all the descriptions are equivalent follows from the fact that the fundamental group of the Lie group $SL_2(\mathbb{R})$ is \mathbb{Z} , so that $SL_2(\mathbb{R})$ has exactly one indecomposable cover of any given finite order. We use here the “modular-form-theoretic” one, and in Section 5 we present its description arising from the theory in [W] and describe the isomorphism between them.

We recall that any element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ defines an action on $\mathcal{H} = \{\tau \in \mathbb{C} | \Im \tau > 0\}$ by $A\tau = \frac{a\tau + b}{c\tau + d}$, and a function on \mathcal{H} , called the *factor of automorphy*, which is given by $j(A, \tau) = c\tau + d$. This gives a 1-cocycle, which explicitly means that as functions on \mathcal{H} we have

$$j(AB, \tau) = j(A, B\tau)j(B, \tau).$$

Then the group $Mp_2(\mathbb{R})$ consists of all pairs (A, φ) with $A \in SL_2(\mathbb{R})$ and φ a holomorphic function on \mathcal{H} satisfying $\varphi(\tau)^2 = j(A, \tau)$. The multiplication is defined by

$$(A, \varphi)(B, \psi) = (AB, \tau \mapsto \varphi(B\tau)\psi(\tau)),$$

and the cocycle condition assures that this is a well-defined product on $Mp_2(\mathbb{R})$. The obvious map $Mp_2(\mathbb{R}) \rightarrow SL_2(\mathbb{R})$ is clearly a double cover.

We define $Mp_2(\mathbb{Z})$ to be the set of elements in $Mp_2(\mathbb{R})$ which lie over $SL_2(\mathbb{Z})$. This is a double cover of $SL_2(\mathbb{Z})$. The algebraic description of $Mp_2(\mathbb{Z})$ is based on the 3 elements

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \quad Z = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right)$$

of $Mp_2(\mathbb{Z})$, where $\sqrt{\tau}$ in S takes values with positive real and imaginary parts. The elements T and S generate $Mp_2(\mathbb{Z})$, Z is of order 4 and generates the center of $Mp_2(\mathbb{Z})$, and the identities $S^2 = (ST)^3 = Z$ hold. We shall use the same notation T , S , and Z for the images of these elements in $SL_2(\mathbb{Z})$, as well as in $Mp_2(\mathbb{Z}_p)$ and in $SL_2(\mathbb{Z}_p)$ for any prime p , without risking confusion. It is clear that $Mp_2(\mathbb{Z})$ is a non-trivial cover of $SL_2(\mathbb{Z})$, since the element $-I$ has order 2 in $SL_2(\mathbb{Z})$ while both its pre-images Z and Z^{-1} are of order 4 in $Mp_2(\mathbb{Z})$. This implies the non-triviality of $Mp_2(\mathbb{R})$ over $SL_2(\mathbb{R})$ as well.

1.3 Weil Representations

To put the Weil representation corresponding to D_M more in the present context, we recall some notions from [W]. In the general context of [W], the finite group D_M is considered as a locally compact Abelian group which is naturally isomorphic to its Pontryagin dual $\widehat{D_M}$ via the \mathbb{Q}/\mathbb{Z} -valued pairing on D_M composed with \mathbf{e} . Since we shall use this construction later for other groups as well, we switch to the general setting, and then apply it to $G = D_M$. Let G be a locally compact Abelian group, and let f be a non-degenerate character of second degree on G (such that in particular $G \cong \widehat{G}$, with \widehat{G} denoting the Pontryagin dual of G). Then the anti-symmetrization of the pairing between G and \widehat{G} gives a symplectic structure on $G \times \widehat{G}$. This allows one to define the symplectic group $Sp(G)$ (in the notation of [W]) of endomorphisms of $G \times \widehat{G}$ which preserves this symplectic structure, and the general theory of [W] now gives a faithful unitary representation of an S^1 -cover of $Sp(G)$ on the space $L^2(G)$. Here and throughout, S^1 denotes the group $\{z \in \mathbb{C} \mid |z| = 1\}$. We note that elements of $Sp(G)$ can be written as 2×2 matrices, having one coordinate in $End(G)$, one in $Hom(G, \widehat{G})$, one in $Hom(\widehat{G}, G)$, and one in $End(\widehat{G})$ (satisfying the symplectic condition). Then the symmetric isomorphism ρ attached to f identifies G with \widehat{G} , allowing us to consider the elements in $Sp(G)$ as having only coordinates in $End(G)$. If the entries of an element in $Sp(G)$ lie in the natural image of \mathbb{Z} in $End(G)$ then “symplecting” is equivalent to “being in $SL_2(\mathbb{Z})$ ”. Hence by restricting to this subgroup we obtain a representation of an S^1 -cover of $SL_2(\mathbb{Z})$. The classical generators T and S in $SL_2(\mathbb{Z})$ can always

be lifted into the elements $T_f = \mathbf{t}_0(f)$ and $\tilde{S}_f = \mathbf{d}'_0(\rho^{-1})$ (in the notation of [W]), and then one lifts Eq. (9) of [W] from $\widehat{Sp}(G)$ to a similar equation in its S^1 -cover containing the factor $\gamma(f)$ (this is how it is defined in [W]), which we call, following [Ra] and others, the *Weil index* of f . By Theorem 2 of [W], the Weil index $\gamma(f)$ of f appears in the (distribution-theoretic) Fourier transform of f , and both in this Fourier transform and in \tilde{S}_f the module of ρ shows up. The normalization of the Haar measure according to the isomorphism ρ is defined such that this module equals unity (i.e., ρ identifies G with this Haar measure with \widehat{G} with the dual measure), which is equivalent to the Fourier Inversion Theorem holding without additional constants. Now, if we further assume that $f(-x) = f(x)$ for any $x \in G$ (we call such f *symmetric*), which is equivalent to the statement that T_f commutes with the parity operator $\tilde{S}_f^2 = \mathbf{d}_0(-1)$, then the lifted Eq. (9) can be written as $(\tilde{S}_f T_f)^3 = \gamma(f) \tilde{S}_f^2$ (without the symmetry condition on f , the left hand side is a bit more complicated). This shows that by defining $S_f = \overline{\gamma(f)} \tilde{S}_f$ we obtain the relation $(S_f T_f)^3 = S_f^2$, and the square of this common element Z_f is scalar multiplication by $\overline{\gamma(f)}^4$. Hence the order of Z_f is twice the order of $\gamma(f)^4$ in S^1 .

Consider now $G = D_M$ with $f = \mathbf{e} \circ q$ (which is quadratic and hence symmetric). Composition with \mathbf{e} defines an isomorphism $\widehat{Hom}(D_M, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \widehat{D_M}$, and ρ is then the isomorphism of D_M with its dual defined by composing this isomorphism with the one arising from the pairing on D_M . The space $L^2(D_M)$ is then $\mathbb{C}[M^*/M]$, having the canonical basis $(e_\gamma)_{\gamma \in M^*/M}$, and we denote this space by V_{ρ_M} . The elements $(e_\gamma)_{\gamma \in M^*/M}$ are mutually L^2 -orthogonal and are all of the same L^2 -norm, and in the normalized Haar measure this common norm is $\frac{1}{\sqrt{\Delta_M}}$. In this case the Weil index is $\gamma(f) = \zeta_8^{sgn(M)}$ (as follows from Theorem 2 of [W], the normalization, and Milgram's formula), hence $\gamma(f)^8 = 1$ and Z_f satisfies $Z_f^4 = Id$. Therefore the map ρ_M which sends $T \mapsto T_f$ and $S \mapsto S_f$ defines a unitary representation of $Mp_2(\mathbb{Z})$ on V_{ρ_M} . Explicitly, this representation is described by the familiar formulae appearing in [B1], [B2], and [Str]:

$$\begin{aligned} \rho_M(T)(e_\gamma) &= \mathbf{e}(\gamma^2/2) e_\gamma, \\ \rho_M(S)(e_\gamma) &= \frac{\zeta_8^{-sgn(M)}}{\sqrt{\Delta_M}} \sum_{\delta \in M^*/M} \mathbf{e}(-(\gamma, \delta)) e_\delta. \end{aligned}$$

The operator $\rho_M(Z)e_\gamma = \zeta_8^{-2sgn(M)} e_{-\gamma}$ indeed satisfies $\rho_M(Z)^4 = Id$, and moreover $\rho_M(Z)^2 = Id$ if and only if the signature (or equivalently the rank) of M is even. The formulae in [Sche] define the dual representation ρ_M^* in the case of even signature.

By considering the complex vector space $V_{\rho_M} = L^2(D_M)$ as the complexification of the real $L^2(D_M)$, we can define the complex conjugate representation $\overline{\rho_M}$ to act on the same space. Then $\overline{\rho_M}$ is isomorphic to the dual representation ρ_M^* (as is always the case when we have a representation space which is a complexification of a real vector space with a bilinear form). This is the

representation considered in [Sche], so that when one wishes to compare our results with those of [Sche], one must take the complex conjugate on one side. It is also clear that $\overline{\rho_M} \cong \rho_M^*$ is isomorphic to $\rho_{M(-1)}$. We further remark that if N is another even lattice then we have

$$\rho_{M \oplus N} \cong \rho_M \otimes \rho_N$$

(with \oplus denoting the *orthogonal* direct sum), an isomorphism which is very similar to the basic idea of the decomposition into p -parts, which we now consider in more detail.

2 Decomposition into p -Parts

In this section we show how ρ_M can be written as the tensor product of the Weil representations of p -adic lattices.

2.1 The p -Parts of D_M as p -adic Discriminant Forms

First we introduce a convenient notation. For any prime p , $\mathbb{Q}_p/\mathbb{Z}_p$ is naturally isomorphic to $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, hence embedded into \mathbb{Q}/\mathbb{Z} . Then we define the character χ_p on \mathbb{Q}_p by

$$\chi_p : \mathbb{Q}_p \rightarrow \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\mathbf{e}} S^1,$$

with kernel \mathbb{Z}_p . When we decompose $\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ and restrict χ_p to \mathbb{Q} (or to \mathbb{Q}/\mathbb{Z}) for each p we find that $\mathbf{e}(x) = \prod_p \chi_p(x)$ (with almost all factors being equal to 1) for all $x \in \mathbb{Q}$. This elementary observation will turn out to be very useful later.

Now, the p -Sylow component of the finite Abelian group $D_M = M^*/M$ is isomorphic to $D_M \otimes_{\mathbb{Z}} \mathbb{Z}_p = M_p^*/M_p = D_{M_p}$, of cardinality $\Delta_{M_p} = p^{v_p(\Delta_M)}$. Hence it is endowed with the $\mathbb{Q}_p/\mathbb{Z}_p$ -valued bilinear form (still denoted (\cdot, \cdot)) and the $\mathbb{Q}_p/\mathbb{Z}_p$ -valued quadratic form q_p coming from those on M . Composing them with χ_p identifies D_{M_p} with its Pontryagin dual $\widehat{D_{M_p}}$, and we denote $\chi_p \circ q_p$ by f_p . Repeating the process as for M yields a representation ρ_{M_p} of $Mp_2(\mathbb{Z})$ on $L^2(D_M) = \mathbb{C}[M_p^*/M_p]$, which we denote $V_{\rho_{M_p}}$. The space $V_{\rho_{M_p}}$ has the natural basis $(e_{\gamma_p})_{\gamma_p \in M_p^*/M_p}$ which is orthogonal of common norm $\frac{1}{\sqrt{\Delta_{M_p}}}$, and on this basis the representation is given explicitly by

$$\begin{aligned} \rho_{M_p}(T)(e_{\gamma_p}) &= \chi_p(\gamma_p^2/2)e_{\gamma_p}, \\ \rho_M(S)(e_{\gamma_p}) &= \frac{\overline{\gamma(f_p)}}{\sqrt{\Delta_{M_p}}} \sum_{\delta_p \in M_p^*/M_p} \chi_p(-(\gamma_p, \delta_p))e_{\delta_p}. \end{aligned}$$

The Weil index satisfies $\gamma(f_p)^4 = 1$ for any odd p (this follows, for example, from the fact that the quadratic form $x^2 + y^2 + z^2 + w^2$ is not equivalent to the reduced norm from the non-trivial quaternion algebra over \mathbb{Q}_p for any odd

p since it represents 0 there—more details on this appear in Section 6), so that for any such p this defines a representation of $SL_2(\mathbb{Z})$. For $p = 2$ we find that $\gamma(f_2)^4 = (-1)^{rk(M)}$, so that this gives a well-defined representation of $SL_2(\mathbb{Z})$ (rather than $Mp_2(\mathbb{Z})$) if and only if the signature (or equivalently the rank) of M is even.

We are going to use congruence subgroups over different rings in what follows, so we introduce some notation. Let R be a ring and N be an ideal of R . If R is topological then we assume that N is closed. We define the subgroup $\Gamma(N, R)$ of $SL_2(R)$ to be the kernel of the canonical map to $SL_2(R/N)$. It is a closed normal subgroup of $SL_2(R)$, and the latter map is surjective. The groups $\Gamma_1(N, R)$ and $\Gamma_0(N, R)$ are defined to be the natural extension of the classical ones as well. As with the definition of the level, principal ideals and their generators are given the same notation here as well, with no possible confusion. If R is the ring of integers in a global field \mathbb{K} then an element in $SL_2(R)$ lies in $\Gamma(N, R)$ (resp. $\Gamma_1(N, R)$, $\Gamma_0(N, R)$) if and only if its images in $SL_2(R_v)$ lie in $\Gamma(N_v, R_v)$ (resp. $\Gamma_1(N_v, R_v)$, $\Gamma_0(N_v, R_v)$) for any place v of \mathbb{K} .

2.2 Decomposition of ρ_M as a Tensor Product

The decomposition we seek is delete $\rho_M = \bigotimes_p \rho_{M_p}$ for some even lattice M . To see why this tensor representation is well-defined (i.e., is essentially a *finite* tensor product), we prove the following

Lemma 2.1. *For any prime p , the following are equivalent: (i) p does not divide Δ_M . (ii) p does not divide N . (iii) The representation ρ_{M_p} is trivial.*

Proof. Assume that (ii) holds. Then we know that $\gamma^2/2 \in \mathbb{Z}_p$ for any $\gamma \in M^*$, hence $\gamma_p^2/2 \in \mathbb{Z}_p$ for any $\gamma_p \in M_p^*$. Thus $(\gamma_p, \delta_p) \in \mathbb{Z}_p$ for every γ_p and δ_p in M_p^* , so that M_p maps isomorphically onto M_p^* . Therefore $\Delta_{M_p} = 1$, which implies (i). Moreover, $\rho_{M_p}(T)$ is trivial, and since we also find that f_p is trivial, we see that $\rho_{M_p}(S)$ is trivial as well. Hence (ii) implies both (i) and (iii). The fact that the kernel of χ_p is precisely \mathbb{Z}_p immediately shows that (iii) implies (ii). Finally, assume (i). Then we have some $k > 0$ such that $p^k \equiv 1 \pmod{\Delta_M}$, hence $p^k \gamma \equiv \gamma \pmod{M}$ for every $\gamma \in M^*$. This implies that $p^{2k} \gamma^2/2 \equiv \gamma^2/2 \pmod{\mathbb{Z}_p}$ for every such γ , which is possible only if $\gamma^2/2$ is itself in \mathbb{Z}_p for every γ . Hence (i) implies (ii), which completes the proof of the lemma. \square

At some points it will be more convenient to consider only those (finitely many) primes p which do not satisfy the conditions of Lemma 2.1. Hence we call these primes *interesting*. It follows from the proof of Lemma 2.1 that if p is not interesting then the trivial representation ρ_{M_p} is 1-dimensional, hence contributes nothing to tensor products. Therefore the tensor product $\bigotimes_p \rho_{M_p}$ is indeed well-defined, and can be considered to be taken only over the finite set of interesting primes. This property applies for all the sums and products over p in the remainder of this Section. In particular, an even lattice has no interesting primes if and only if it is unimodular, and indeed in this case the representation ρ_M is 1-dimensional and trivial, in correspondence with the following paragraph.

All this holds only under the assumption that the lattice M is even—see Section 9 for the odd lattice case.

Since $D_M = \bigoplus_p D_{M_p}$, we clearly have $V_{\rho_M} = \bigotimes_p V_{\rho_{M_p}}$, and obviously $\Delta_M = \prod_p \Delta_{M_p}$ as well. In particular, ρ_M and $\bigotimes_p \rho_{M_p}$ act on the same space. Moreover, for any $\gamma \in D_M$ and prime p , denote its image in $D_{M_p} = D_M \otimes_{\mathbb{Z}} \mathbb{Z}_p$ by γ_p (this may be non-zero only for interesting p). Then in the direct sum decomposition $D_M = \bigoplus_p D_{M_p}$ we have that $\gamma = \sum_p \gamma_p$, and summing over all $\delta \in D_M$ gives the same answer as summing over all $\delta_p \in D_{M_p}$ for all (interesting) primes p . Moreover, for any γ and δ in D_M we have that (γ_p, δ_p) and $\frac{\gamma_p^2}{2}$ are the $\mathbb{Q}_p/\mathbb{Z}_p$ -images of (γ, δ) and $\frac{\gamma^2}{2}$, hence are the unique elements a_p and b_p in $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ such that $(\gamma, \delta) = \sum_p a_p$ and $\frac{\gamma^2}{2} = \sum_p b_p$. This means that the decomposition $D_M = \bigoplus_p D_{M_p}$ is in fact an *orthogonal* decomposition. It follows that $\rho_M(T)$ coincides with $\bigotimes_p \rho_{M_p}(T)$, and up to the Weil indices $\rho_M(S)$ coincides with $\bigotimes_p \rho_{M_p}(S)$. Now, both are representations of $Mp_2(\mathbb{Z})$ in which the relation $(ST)^3 = S^2$ holds, and as we have seen implicitly in Section 1, there is only one possibility to choose the S^1 -scalar multiplying S in order for this relation to hold. This implies that the two representations $\bigotimes_p \rho_{M_p}$ and ρ_M must indeed coincide.

In fact, the statement $\rho_M(S) = \bigotimes_p \rho_{M_p}(S)$ is equivalent to the Weil Reciprocity Law, which in our conventions states that $\gamma(f_{\mathbb{R}}) = \prod_p \gamma(f_{\mathbb{Q}_p})$. In order to see this, we note that for any prime p , Section 27 of [W] evaluates $\gamma(f_{\mathbb{Q}_p})$ using a Gauss sum, which can be taken to be the one defining $\gamma(f_p)$. Hence $\gamma(f_{\mathbb{Q}_p}) = \gamma(f_p)$ for every prime p , so that the right hand side of the Weil Reciprocity Law is the complex conjugate of the S^1 -multiplier appearing in $\bigotimes_p \rho_{M_p}(S)$. On the other hand, Section 26 of [W] evaluates the Weil index $\gamma(f_{\mathbb{R}})$ appearing as the left hand side of the Weil Reciprocity Law as the complex conjugate of the number $\zeta_8^{-\text{sgn}(M)}$ appearing in $\rho_M(S)$. Hence our proof that $\rho_M(S) = \bigotimes_p \rho_{M_p}(S)$ gives another proof of the Weil Reciprocity Law.

2.3 $Mp_2(\mathbb{Z}_p)$ Acting on the p -part

Let us change our point of view on M_p^*/M_p , and see how we can consider the representation ρ_{M_p} as the restriction to $Mp_2(\mathbb{Z})$ of a representation of $Mp_2(\mathbb{Z}_p)$. The locally compact group we now consider is the vector space $M_{\mathbb{Q}_p}$. The bilinear form identifies it with its dual vector space (also topologically), and composing linear functionals with χ_p identifies it also with its Pontryagin dual $\widehat{M_{\mathbb{Q}_p}}$. Then we have an action of an S^1 -cover of the symplectic group $Sp(M_{\mathbb{Q}_p})$ on the space $L^2(M_{\mathbb{Q}_p})$ by the process described in Section 1, under which the dense subspace $\mathcal{S}(M_{\mathbb{Q}_p})$ of Schwartz functions on $M_{\mathbb{Q}_p}$ is invariant (see Sections 11–12 of [W]). Moreover, Section 35 of [W] shows that this representation is continuous in the strong topology on the group of unitary operators on $L^2(M_{\mathbb{Q}_p})$, and can be restricted to a representation of a double cover $Mp(M_{\mathbb{Q}_p})$ of $Sp(M_{\mathbb{Q}_p})$. In particular we get a representation of a double cover $Mp(\mathbb{Q}_p)$ of $SL(\mathbb{Q}_p)$ on $L^2(M_{\mathbb{Q}_p})$ (This point is discussed further, in greater generality, in Section

5). This representation can be further restricted to a representation of a double cover $Mp(\mathbb{Z}_p)$ of $SL(\mathbb{Z}_p)$, and the latter representation in fact splits over $SL(\mathbb{Z}_p)$ in many cases—see Section 6.

In the case of a p -adic vector space, a function on $M_{\mathbb{Q}_p}$ is in $\mathcal{S}(M_{\mathbb{Q}_p})$ if its support is contained in a sufficiently large \mathbb{Z}_p -submodule of $M_{\mathbb{Q}_p}$ and it is constant on cosets of a small enough \mathbb{Z}_p -submodule of $M_{\mathbb{Q}_p}$. In particular, we consider the space of functions which are supported in M_p^* and are constant on cosets of M_p . Moreover, if we identify, for $\gamma_p \in D_{M_p}$, the canonical basis element e_{γ_p} of $V_{\rho_{M_p}}$ with the characteristic function of the coset $M_p + \gamma_p$, then we obtain a canonical isomorphism between $V_{\rho_{M_p}}$ and the (finite-dimensional) subspace of $\mathcal{S}(M_{\mathbb{Q}_p})$ just described. In fact, every element of $\mathcal{S}(M_{\mathbb{Q}_p})$ is contained in a subspace of this form if we take the p -adic lattice M_p to be small enough. We now prove

Lemma 2.2. *The subspace of $\mathcal{S}(M_{\mathbb{Q}_p})$ described above is stable under the action of $Mp_2(\mathbb{Z}_p)$. Moreover, under the identification just described, the restriction of the representation of $Mp_2(\mathbb{Z}_p)$ on this space to $Mp_2(\mathbb{Z})$ is the p -part ρ_{M_p} of the original Weil representation ρ_M .*

We remark that the first statement of Lemma 2.2 will be proved directly via a detailed calculation in Section 7. However, it also follows from the second claim, as the proof below shows.

Proof. We repeat the process of Section 1 for the locally compact Abelian group $M_{\mathbb{Q}_p}$. The normalized Haar measure on $M_{\mathbb{Q}_p}$ attains $\frac{1}{\sqrt{\Delta_{M_p}}}$ on M_p , and we have seen that $\gamma(f_{\mathbb{Q}_p}) = \gamma(f_p)$. Thus T_f multiplies every function $\Phi \in \mathcal{S}(M_{\mathbb{Q}_p})$ by the function $x \mapsto f_{\mathbb{Q}_p}(x) = \chi_p(\frac{x^2}{2})$, and S_f takes Φ to its Fourier transform multiplied by $\overline{\gamma(f_p)}$. Take $\Phi = e_{\gamma_p}$, i.e., Φ is the characteristic function of $M_p + \gamma_p$ for some $\gamma_p \in D_{M_p}$. Then for every x in that coset the equality $\frac{x^2}{2} \equiv \frac{\gamma_p^2}{2} \pmod{\mathbb{Z}_p}$ holds, so that T_f multiplies $\Phi = e_{\gamma_p}$ by $f_p(\gamma_p) = \chi_p(\frac{\gamma_p^2}{2})$. On the other hand, the Fourier transform of a function $\Phi \in \mathcal{S}(M_{\mathbb{Q}_p})$ is defined by

$$\widehat{\Phi}(x) = \int_{M_{\mathbb{Q}_p}} \Phi(y) \overline{\chi_p((x, y))} dy,$$

which for $\Phi = e_{\gamma_p}$ just gives $\overline{\chi_p((x, \gamma_p))} \int_{M_p} \overline{\chi_p((x, u))} du$. Now, the integral vanishes for $x \notin M_p^*$ and gives $\frac{1}{\sqrt{\Delta_{M_p}}}$ for $x \in M_p^*$, and in the latter case the coefficient $\overline{\chi_p((x, \gamma_p))}$ is constant on cosets of M_p in M_p^* . Putting in the Weil index $\gamma(f_{\mathbb{Q}_p}) = \gamma(f_p)$ completes the proof of the second assertion, since T and S generate $Mp_2(\mathbb{Z})$. It follows that the action of $Mp_2(\mathbb{Z})$ preserves the space in question, and since the representation is continuous and $Mp_2(\mathbb{Z})$ is dense in $Mp_2(\mathbb{Z}_p)$, the first assertion follows as well. This proves the lemma. \square

Using this identification we consider ρ_{M_p} as a representation of $Mp_2(\mathbb{Z}_p)$, which we continue to denote ρ_{M_p} . We further remark that since $\gamma(f_p)^4 = 1$

for any odd p , ρ_{M_p} this is in fact a representation of $SL_2(\mathbb{Z})$, and Lemma 2.2 shows that it is a representation of $SL_2(\mathbb{Z}_p)$. This observation will simplify some technical arguments below.

It is worth mentioning at this point that one can obtain ρ_M as a representation arising from the Weil representation on $\mathcal{S}(M_{\mathbb{R}})$ as well, using a different, more delicate argument. This observation can be used to show that $\gamma(f) = \gamma(f_{\mathbb{R}})$, for example. We describe the argument briefly, even though we do not use it in this paper. For any $\Phi \in \mathcal{S}(M_{\mathbb{R}})$ define the function $\bar{\Phi} \in \mathcal{S}(M_{\mathbb{R}}/M)$ by $\bar{\Phi}(x) = \sum_{\lambda \in M} \Phi(x + \lambda)$ (This is the operator Z given in Eq. (20) of [W], with $G = M_{\mathbb{R}}$ and $\Gamma = M$, restricted to the character x^* being 0.) Restricting this function to M^*/M gives an element in V_{ρ_M} (under the usual identification). The actions of $T_{f_{\mathbb{R}}}$ and $S_{f_{\mathbb{R}}}$ on $\mathcal{S}(M_{\mathbb{R}})$ commute with this map to V_{ρ_M} and the representation operators $\rho_M(T)$ and $\rho_M(S)$, up to the difference in scalars $\gamma(f)$ and $\gamma(f_{\mathbb{R}})$. Since the element of S^1 required to preserve the relation $(ST)^3 = S^2$ is unique, we obtain that $\gamma(f) = \gamma(f_{\mathbb{R}}) = \zeta_8^{sgn(M)}$, as required. A similar argument is employed in Section 2 of [Sh], and can also be used to prove Milgram's formula, as is implicitly done in Corollary 4.2 of [B1]. We note that such an argument can be applied also for $M_{\mathbb{Q}_p}$, but the one given in Lemma 2.2 is simpler.

3 Factoring of the Weil Representation

In this Section we use the decomposition from Section 2 to prove the factoring claim, namely Theorem 3.2. In fact, all the results in this Section will be proven again in Section 8, but the proof presented here demonstrates the use of the decomposition before numerous technical details have to be presented.

3.1 Closed Normal Subgroups of $SL_2(\mathbb{Z}_p)$

The essential ingredient of the proof of the factoring claim is the following

Lemma 3.1. *For every prime number p , the minimal closed normal subgroup of $SL_2(\mathbb{Z}_p)$ containing T^N is $\Gamma(N, \mathbb{Z}_p)$.*

Proof. Let Γ denote the minimal normal closed subgroup in question. Since clearly $\Gamma(N, \mathbb{Z}_p)$ is a normal closed subgroup of $SL_2(\mathbb{Z}_p)$ containing T^N , we have $\Gamma \leq \Gamma(N, \mathbb{Z}_p)$. We must now show the reverse inclusion. For any $b \in \mathbb{Z}$ such that $N|b$ we know that $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ is in Γ (as a power of T^N), and this extends by continuity to $b \in \mathbb{Z}_p$ such that $N|b$ since Γ is closed. Moreover, every element of the form $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ with $c \in \mathbb{Z}_p$ divisible by N lies in Γ as the conjugate of $\begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}$ via the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, since Γ is normal.

Now, if p is interesting, i.e., $p|N$, we know that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N, \mathbb{Z}_p)$ then $d \in \mathbb{Z}_p^*$ (as it satisfies $d \equiv 1 \pmod{p^{v_p(N)}}$ and $v_p(N) > 0$). Let $\tilde{\Gamma}$ be the

subgroup of $SL_2(\mathbb{Z}_p)$ consisting of these matrices such that $b \equiv c \equiv 0 \pmod{N}$ and $a \equiv d \equiv 1 \pmod{N^2}$, and note that this imply $\frac{a-1}{N} \equiv -\frac{d-1}{N} \pmod{N}$. It is clear that $\Gamma(N^2, \mathbb{Z}_p) \leq \tilde{\Gamma} \leq \Gamma(N, \mathbb{Z}_p)$, and in particular $\tilde{\Gamma}$ is a congruence subgroup. We now show that $\tilde{\Gamma} \leq \Gamma$. Indeed, every matrix in $\tilde{\Gamma}$ can be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{b-N}{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{d-1}{N} & 1 \end{pmatrix} \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1-d+Nc}{Nd} & 1 \end{pmatrix},$$

where the fact that $d \in \mathbb{Z}_p^*$ and $a = \frac{1+bc}{d}$ and the congruence conditions show that all the multipliers are elements of Γ by the preceding paragraph. We note that if p is not interesting then the same argument shows that any matrix with invertible d is in Γ .

Now, for any $k \in \mathbb{Z}_p$, the conjugation of $\begin{pmatrix} 1 & 0 \\ kN & 1 \end{pmatrix} \in \Gamma$ by T gives the element $\begin{pmatrix} 1+kN & -kN \\ kN & 1-kN \end{pmatrix}$ (which is thus also in Γ), and we can write general element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N, \mathbb{Z}_p)$ as

$$\begin{pmatrix} a - kN(a+b) & b + kN(a+b) \\ c - kN(c+d) & d + kN(c+d) \end{pmatrix} \begin{pmatrix} 1+kN & -kN \\ kN & 1-kN \end{pmatrix}.$$

Now, if p is interesting then the right multiplier is in Γ for any k , and if we choose k such that $k \equiv \frac{a-1}{N} \equiv -\frac{d-1}{N} \pmod{N}$ then the left multiplier is also in Γ . On the other hand, if p is not interesting and d is not invertible, then c and $c+d$ are invertible. Since N is also invertible in \mathbb{Z}_p (as p is not interesting), taking any $k \in \mathbb{Z}_p^*$ here gives a presentation of our matrix as a product of two elements in Γ . This completes the proof of the lemma. \square

We note that the advantage of working over \mathbb{Z}_p rather than over \mathbb{Z} in Lemma 3.1 is the fact that $d \in \mathbb{Z}_p^*$ for interesting p while d is usually not invertible in \mathbb{Z} . This difference is essential, since the existence of non-congruence subgroups in $SL_2(\mathbb{Z})$ implies that the statement of Lemma 3.1 does not hold over \mathbb{Z} .

3.2 The Factoring Claim

We now present our proof of the factoring claim, i.e.,

Theorem 3.2. *The Weil representation ρ_M factors through a double cover $Mp_2(\mathbb{Z}/N\mathbb{Z})$ of $SL_2(\mathbb{Z}/N\mathbb{Z})$, and it factors further through $SL_2(\mathbb{Z}/N\mathbb{Z})$ itself if and only if the rank of M is even.*

Proof. It is clear that the element $\rho_M(T^N)$ (with $T \in Mp_2(\mathbb{Z})$ now) is trivial (either as $\bigotimes_p \rho_{M_p}(T^N)$ or directly), and in particular $T^N \in \ker \rho_{M_p}$ for every prime p . Since ρ_{M_p} is continuous, the minimal normal closed subgroup of $Mp_2(\mathbb{Z}_p)$ containing T^N is contained in $\ker \rho_{M_p}$. By adding Z^2 to this subgroup we obtain the (inverse image of the) minimal normal closed subgroup of

$SL_2(\mathbb{Z}_p)$, which is $\Gamma(N, \mathbb{Z}_p)$ by Lemma 3.1. Moreover, we have seen that ρ_{M_p} is a representation of $SL_2(\mathbb{Z}_p)$ for odd p (i.e., $\rho_{M_p}(Z^2)$ is trivial), hence $\ker \rho_{M_p}$ contains the inverse image of $\Gamma(N, \mathbb{Z}_p)$ for any odd p . For $p = 2$ we have that $\ker \rho_{M_2}$ contains a subgroup of $Mp_2(\mathbb{Z}_2)$ which lies over $\Gamma(N, \mathbb{Z}_2)$. Furthermore, $\rho_M(Z^2) = \bigotimes_p \rho_{M_p}(Z^2)$ is $\rho_{M_2}(Z^2)$ by the same argument, so that $\gamma(f_2)^4$ (which is $(-1)^{\text{oddity}(M)}$ in the notation of [Sche] and [Str]) is the same as $(-1)^{rk(M)}$. Thus $\ker \rho_{M_2}$ contains a double cover (resp. a lift) of $\Gamma(N, \mathbb{Z}_2)$ exactly when the rank of M is even (resp. odd).

Now let γ be an element of $Mp_2(\mathbb{Z})$ lying over an element of $\Gamma(N)$. This is equivalent to the image of γ in $Mp_2(\mathbb{Z}_p)$ lying over $\Gamma(N, \mathbb{Z}_p)$ for all p . Then the previous paragraph shows that $\rho_{M_p}(\gamma)$ is trivial for any odd p , and up to multiplying γ by the element Z^2 separating $Mp_2(\mathbb{Z})$ from $SL_2(\mathbb{Z})$, $\rho_{M_2}(\gamma)$ is trivial as well. More accurately, $\rho_{M_2}(\gamma)$ is either trivial or equals $\rho_{M_2}(Z^2)$. Therefore $\rho_M(\gamma) = \bigotimes_p \rho_{M_p}(\gamma)$ is also either trivial or equals $\rho_{M_2}(Z^2) = \rho_M(Z^2)$. Hence if $rk(M)$ is even then $\rho_M(\gamma)$ is trivial in any case, and we find that ρ_M factors through $SL_2(\mathbb{Z})/\Gamma(N) = SL_2(\mathbb{Z}/N\mathbb{Z})$. On the other hand, if $rk(M)$ is odd then the intersection of $\ker \rho_M$ with the inverse image of $\Gamma(N)$ in $Mp_2(\mathbb{Z})$ gives a normal subgroup of $Mp_2(\mathbb{Z})$ which maps isomorphically onto $\Gamma(N)$ and whose ρ_M -action is trivial. Therefore ρ_M factors through the corresponding quotient, which is a double cover $Mp_2(\mathbb{Z}/N\mathbb{Z})$ of $SL_2(\mathbb{Z}/N\mathbb{Z})$. This proves the Theorem. \square

We shall see in Section 8 that the kernel of ρ_M is exactly this lift or inverse image of $\Gamma(N)$, except in some particular cases where $\ker \rho_M$ contains this group as a subgroup of index 2.

The proof of Theorem 3.2 implies that if N is the level of a lattice of odd rank then $\Gamma(N)$ can be lifted into a normal subgroup of $Mp_2(\mathbb{Z})$. In particular, since the even 1-dimensional lattice spanned by an element x with $x^2 = 2$ has level 4, this can be done for $\Gamma(4)$. Lifting $\Gamma(N)$ in this way is possible only for $4|N$, since Theorem 3.2 has the following

Corollary 3.3. *If the rank of the even lattice M is odd then the level N of M is divisible by 4.*

Proof. We know that the kernel of ρ_{M_2} contains a subgroup of $Mp_2(\mathbb{Z}_2)$ which lies over $\Gamma(N, \mathbb{Z}_2) = \Gamma(2^{v_2(N)}, \mathbb{Z}_2)$. Now, if $4 \nmid N$ then $v_2(N) \leq 1$ and $\ker \rho_{M_2}$ contains a subgroup lying over $\Gamma(2, \mathbb{Z}_2)$. But $-I \in \Gamma(2, \mathbb{Z}_2)$, and the elements lying over it are Z or $Z^3 = Z^{-1}$. Since $Z^2 = Z^{-2}$ is then the square of an element in $\ker \rho_M$, we obtain $\rho_M(Z^2) = 1$. But we have seen that $\rho_M(Z^2) = (-1)^{rk(M)}$, whence the corollary. \square

In Section 8 we shall give an explicit description of the lift of $\Gamma(N)$ into $Mp_2(\mathbb{Z})$ obtained as $\ker \rho_M$ for some even lattice M of level N and odd rank. Moreover, Section 8 shows that the lift of $\Gamma(N)$ is unique, i.e., independent of the lattice M of level N we use in order to obtain it, and is the restriction of the lift of $\Gamma(4)$.

We remark that the first paragraph of the proof of Theorem 3.2 reproves (ii) \Rightarrow (iii) of Lemma 2.1. Indeed, if p does not divide N then $\Gamma(N, \mathbb{Z}_p)$ is the full group $SL_2(\mathbb{Z}_p)$. Since Corollary 3.3 shows that if N is odd then $\rho_{M_2}(Z^2)$ is trivial, we see that indeed ρ_{M_p} must be trivial for any such p .

4 p -adic Lattices and their Gauss Sums

In this Section we skim through Jordan decompositions of p -adic lattices, together with the corresponding Gauss sums and Weil indices. Then we evaluate a more general Gauss Sum, corresponding to such a lattice and to an element of $SL_2(\mathbb{Z}_p)$. This result which will be needed in Section 8. We note that our treatment of p -adic lattices is related to the discussion in [Sche] and [Str] about discriminant forms, the main difference being the unimodular parts: these may appear in the lattices but vanish in their discriminant forms.

4.1 The Quadratic Reciprocity Law and Related Objects

We begin with some notation. For any odd K we adopt the classical notation, and define ε_K to be 1 if $K \equiv 1(4)$ and i if $K \equiv 3(4)$, and $\varepsilon(K)$ to be the image of $\frac{K-1}{2}$ in $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. In addition we define, for any non-zero rational (or real) number x , the element $\sigma(x) \in \mathbb{F}_2$ such that $\text{sgn}(x) = (-1)^{\sigma(x)}$. We extend the Legendre symbol $(\frac{x}{y})$ also for negative odd y by defining $(\frac{x}{y}) = (\frac{x}{|y|})$. We remark that this is different from the Kronecker extension used in [Str], [B3], and [Sh], which in our notation is given by $(\frac{x}{|y|})(-1)^{\sigma(x)\sigma(y)}$. Although the latter formula preserves the equality $(\frac{-1}{y}) = (-1)^{\varepsilon(y)}$ (while our definition yields $(\frac{-1}{y}) = (-1)^{\varepsilon(y)+\sigma(y)}$), our formula has the advantage that $(\frac{x}{y})$ depends only on the value of x modulo y . Moreover, our convention extends further to the *quadratic power residue symbol* defined over more general number fields in page 24 of [Ge]. Both extensions are multiplicative in x and in y , and in both extensions the quadratic reciprocity law extends to the statement that $(\frac{x}{y})(\frac{y}{x}) = (-1)^{\varepsilon(x)\varepsilon(y)+\sigma(x)\sigma(y)}$ (see also Eq. (5.5) of [Str]). We note, in relation with Section 5, that $(-1)^{\sigma(x)\sigma(y)}$ is the same as the Hilbert symbol $(x, y)_{\mathbb{R}}$ over the field of real numbers.

We further note that while the Legendre symbol $(\frac{x}{y})$ is defined for x and y in \mathbb{Z} with y odd, in the special cases $(\frac{2^k}{y})$ and $(\frac{x}{p^k})$ with odd p and $k \geq 0$ we can take, more generally, $y \in \mathbb{Z}_2$ and $x \in \mathbb{Z}_p$. Indeed, these are defined for elements in \mathbb{Z} by the residue of y modulo 8 (and then $(\frac{2^k}{y})$ is symmetric in the sign of y) and of x modulo p , hence can be extended to the asserted domains of definition by continuity. The power k can be taken out of the symbol by multiplicativity. Note that while the symbol vanishes for $y \in 2\mathbb{Z}_2$ or $x \in p\mathbb{Z}_p$ if $k \geq 1$, when $k = 0$ the symbol equals 1 for any $y \in \mathbb{Z}_2$ and $x \in \mathbb{Z}_p$, invertible or not.

We shall also make use of the following formula, which holds for any odd

number x :

$$\left(\frac{2}{x}\right)\varepsilon_x = \zeta_8^{1-x}. \quad (1)$$

This formula appears as Eq. (5.4) of [Str], and we extend it by continuity to $x \in \mathbb{Z}_2^*$. The proof is obtained by checking the 4 possibilities of x modulo 8.

4.2 Jordan Decompositions and Jordan Components

It is well-known that any p -adic lattice M is isomorphic to an orthogonal direct sum $\bigoplus_{e=0}^k M_e(p^e)$, where M_e is unimodular for any e and $M_e(p^e)$ denotes multiplication of the bilinear form by p^e . Moreover, the lattice $M_e(p^e)$ is of the form $q^{\varepsilon n}$ if $p \neq 2$ and $q_t^{\varepsilon n}$ or $q_{II}^{\varepsilon n}$ (the latter appears only with even n) if $p = 2$. In these symbols $q = p^e$, n is the rank of M_e , and $\varepsilon = \pm$ is $(\frac{\text{disc}(M_e)}{p})$ for odd p and $(\frac{2}{\text{disc}(M_e)})$ for $p = 2$. If $p = 2$ then an index t denotes the image of the trace of a diagonal form of M_e in $\mathbb{Z}_2/8\mathbb{Z}_2 = \mathbb{Z}/8\mathbb{Z}$ (this can be seen to be independent of the diagonal form chosen) and an index II denotes that no odd entries appear on the diagonal of a matrix representing M_e (i.e., M_e is an even lattice). The index t must be of the same parity as n , and for small values of n not all the combinations of $t \equiv n \pmod{2}$ and ε can appear: For $n = 1$ we know that $t = \pm 1$ implies $\varepsilon = +$ while $t = \pm 5$ implies $\varepsilon = -$ (since then $\varepsilon = (\frac{2}{t})$), while for $n = 2$ we have that $t = 0$ implies $\varepsilon = +$ while $t = 4$ implies $\varepsilon = -$ (since in $\mathbb{Z}/8\mathbb{Z}$ no sum of two ± 1 or two ± 5 can give 4, and no sum of ± 1 and ± 5 can give 0). Since for any p we have $p^0 = 1$, we denote the unimodular part M_0 by $(p^0)^{\varepsilon n}$ etc. in cases where confusion as to the prime under consideration may arise. For odd p this decomposition is unique in the sense that direct sums with different invariants are never isomorphic (this has been shown by many authors; for a recent generalization to lattices over complete valuation rings of arbitrary rank see [Z1]). For $p = 2$ different decomposed forms may give isomorphic 2-adic lattices, but it is known precisely when this happens (see [J], with some remarks in [Z1]). As already noted, any p -adic lattice with odd p is even, and a 2-adic lattice is even if and only if M_0 is even, i.e., M_0 is of the form $1_{II}^{\varepsilon n}$ with even n (or is trivial, i.e., of rank 0).

Any decomposition of M as $\bigoplus_{e=0}^k M_e(p^e)$ with M_e unimodular for every e is called a *Jordan decomposition*, and the sublattices $M_e(p^e)$ (or equivalently $q^{\varepsilon n}$, $q_t^{\varepsilon n}$, or $q_{II}^{\varepsilon n}$) are called the *components* of the decomposition, or, more abstractly, *Jordan components*. In the direct sum of two Jordan components with the same q (for $q = 1$ this means, of course, p^0 with the same prime p) we find that the ranks are added and the signs are multiplied. For $p = 2$ the index t is added, II is considered to be 0 when added to some t , and the sum of two II indices remains II . A Jordan component is called *indecomposable* if it cannot be presented as the orthogonal direct sum of smaller p -lattices (or equivalently, as the orthogonal direct sum of smaller Jordan components). It is easily seen that the only indecomposable Jordan components are $q^{\varepsilon 1}$ if $p \neq 2$ and $q_t^{\varepsilon 1}$ and $q_{II}^{\varepsilon 2}$ if $p = 2$.

4.3 The Classical Gauss Sums for p -adic Lattices

For any even p -adic lattice one can define its p -excess if p is odd and its oddity if $p = 2$. These are numbers in $\mathbb{Z}/8\mathbb{Z}$, which are described for the Jordan components of discriminant forms in [Sche] and [Str] (and are called signature in [B3]). They can be equally described for the p -adic lattices themselves. In both points of view they are additive with respect to orthogonal direct sums. In both [Sche] and [Str] the roots of unity $\gamma_p = \zeta_8^{-p-\text{excess}}$ for odd p and $\gamma_2 = \zeta_8^{\text{oddity}}$ are defined, and knowing the factors γ_p for all p is equivalent to knowing the p -excess and the oddity. Now let M be an even p -adic lattice, f the (quadratic) character of second degree on D_M defined by $\lambda \mapsto \chi_p(\frac{\lambda^2}{2})$, and $f_{\mathbb{Q}_p}$ the induced (quadratic) character of second degree on $M_{\mathbb{Q}_p}$. Then we already found that $\gamma(f) = \gamma(f_{\mathbb{Q}_p})$, and this value coincides with the root of unity γ_p of [Sche] and [Str]. We call this common root of unity the *Weil index of M* , and denote it by $\gamma(M)$. This root of unity appears in the evaluation of the Gauss sum corresponding to M which is used in Section 27 of [W]. It is evaluated in the following

Proposition 4.1. *The Weil index of a p -adic Jordan component $q^{\varepsilon n}$ with odd p is $\varepsilon^{v_p(q)} \zeta_8^{n(1-q)}$. The Weil index of a 2-adic Jordan component $q_{t/II}^{\varepsilon n}$ is $\varepsilon^{v_2(q)} \zeta_8^t$, where for the index II we take $t = 0$. Moreover, for any even p -adic lattice M the equality $\sum_{\eta \in M^*/M} \chi_p(\frac{\eta^2}{2}) = \gamma(M) \sqrt{\Delta_M}$ holds.*

This is essentially Proposition 3.1 of [Sche] or Lemma 3.1 of [Str]. Since they work with discriminant form and we work with lattices, we give the proof. We could have used more general quadratic Gauss sums, like in Lemmas 3.6., 3.7, and 3.8 of [Str], but here we show how the classical Gauss sums and a few explicit evaluations suffice to give the desired result.

Proof. We note that the Weil index, the Gauss sum, and the cardinality Δ_M are all multiplicative with respect to orthogonal direct sums. Moreover, the asserted formulae for the Weil index of the Jordan components is also multiplicative with respect to orthogonal direct sums, as can be seen from the behavior of the symbols of the Jordan components under this operation. Hence it suffices to verify all the assertions for the indecomposable Jordan components. Moreover, from Section 27 of [W] we know that the Weil index can be calculated by the Gauss sum divided by its absolute value. Hence the problem reduces to evaluating the Gauss sum, verifying that its absolute value is $\sqrt{\Delta_M}$, and finding the root of unity involved.

In order to reduce the possible values of q to only a small finite number of small powers of p we use the p -adic analog of Lemma 1 in Appendix 4 of [MH]. The argument is simple, and is independent of the indecomposability of the Jordan component. If M is a Jordan component with some $q = p^e$, then consider the sublattice $L = pM$ of M . The index $[M : L]$ is finite and equals p^n (and $L^* = \frac{1}{p} M^*$ implies $[L^* : M^*] = p^n$ and $\Delta_L = p^{2n} \Delta_M$), and as a Jordan component L has the same parameters as M but with q replaced by $p^2 q$. For the evaluation of the Gauss sum corresponding to L we can decompose

$\sum_{\eta \in L^*/L} \chi_p\left(\frac{\eta^2}{2}\right)$ as $\sum_{\eta \in L^*/M} \sum_{\lambda \in M/L} \chi_p\left(\frac{(\eta+\lambda)^2}{2}\right)$, and note that $\frac{\lambda^2}{2}$ is in \mathbb{Z}_p and the sum $\sum_{\lambda \in M/L} \chi_p((\eta, \lambda))$ vanishes if η is not in M^* and gives the index $[M : L] = p^n$ (which is the ratio $\frac{\sqrt{\Delta_M}}{\sqrt{\Delta_L}}$) for $\eta \in M^*$. Thus, if the assertion holds for M then it also holds for L as well, since the asserted value of the γ factor is easily seen to be the same for q and p^2q (to see this, note that if $p = 2$ then t is not affected, while for $p \neq 2$ we have that $p^2q \equiv q \pmod{8}$). The last assertion also follows from the fact that $\gamma(M)$ depends only on $M_{\mathbb{Q}_p}$ and clearly $M_{\mathbb{Q}_p} = L_{\mathbb{Q}_p}$ as \mathbb{Q}_p -lattices. It only remains to verify the assertion in the case where q is the minimal power of p for which the (indecomposable) Jordan component is still an even lattice, and in the case where q is the consecutive power of p .

Let us now verify these assertions. For odd p we have to check the sum for 1^{ε_1} and p^{ε_1} , and for $p = 2$ we have to check $1_{II}^{\varepsilon_1^2}$ and $2_{II}^{\varepsilon_1^2}$ for the even Jordan components and $2_i^{\varepsilon_1}$ and $4_i^{\varepsilon_1}$ for the odd ones (recall that we are restricted to work with even lattices!). If $q = 1$ then the sum is trivially 1, which is the required value. For odd p we find that $(p^{\varepsilon_1})^*/p^{\varepsilon_1}$ is $(\mathbb{Z}/p\mathbb{Z})x$ where $x^2 = \frac{u}{p}$ with $u \in \mathbb{Z}_p^*$ which satisfies $\left(\frac{2u}{p}\right) = \varepsilon$. Hence the classical result of Gauss implies that the sum in question is $\varepsilon\left(\frac{2}{p}\right)\varepsilon_p\sqrt{p}$, which is the asserted value by Eq. (1). Similarly, $(2_i^{\varepsilon_1})^*/2_i^{\varepsilon_1}$ is $(\mathbb{Z}/2\mathbb{Z})x$ with $x^2 = \frac{t}{2}$, hence the total value is $1 + i^t$. Since $t = \pm 1$ implies $\varepsilon = +$ and $t = \pm 5$ implies $\varepsilon = -$ we see that in each case this gives the desired value $\varepsilon\zeta_8^t\sqrt{2}$. Continuing in the same manner, $(4_i^{\varepsilon_1})^*/4_i^{\varepsilon_1}$ is $(\mathbb{Z}/4\mathbb{Z})x$ with $x^2 = \frac{t}{4}$, and since t is odd and $3^2 \equiv 1 \pmod{8}$ we find that $1 + \zeta_8^t - 1 + \zeta_8^t = \zeta_8^t\sqrt{4}$ has the desired value. We are left with $2_{II}^{\varepsilon_1^2}$, and we know that $(2_{II}^{\varepsilon_1^2})^*/2_{II}^{\varepsilon_1^2}$ can be spanned over \mathbb{F}_2 by two elements x and y such that $(x, y) = \frac{1}{4}$ where $x^2 = y^2$ is 0 if $\varepsilon = +$ and is $\frac{1}{2}$ if $\varepsilon = -$. The identities $1 + 1 + 1 - 1 = +\sqrt{4}$ and $1 - 1 - 1 - 1 = -\sqrt{4}$ imply the assertion for this case as well. This completes the proof of the proposition. \square

Examining the oddity and p -excess that are defined in [Sche], [Str], and [B3], one can easily verify that as elements in $\mathbb{Z}/8\mathbb{Z}$, $\frac{8}{2\pi i} \log \gamma(M)$ equals the oddity of M for $p = 2$ and equals the negative of the p -excess of M for odd p . For the Jordan component one notes that the ε factors account for the number k (called antisquare in [B3]) which is defined in these references by distinguishing different cases. Since all the numbers involved are multiplicative with respect to orthogonal direct sums, this completes the verification. It follows that the oddity formula, i.e., the formula relating the oddity, p -excess, and signature of a \mathbb{Z} -lattice (or discriminant form) is just the Weil Reciprocity Law when expressed in our terminology.

4.4 More General Gauss Sums

We are interested in what becomes of the Weil index of a p -adic lattice when we multiply the bilinear form on the lattice by some p -adic integer. Clearly, if M has a Jordan decomposition as above, then $M(c)$ has the Jordan decomposition

$\bigoplus_{e=0}^k M_e(p^e c)$, so that it suffices to restrict our attention to a Jordan component $q^{\varepsilon n}$ or $q_{t/II}^{\varepsilon n}$. Writing $c = p^{v_p(c)} c_p$ with $c_p \in \mathbb{Z}_p^*$ separates the problem to powers of p and elements in \mathbb{Z}_p^* , and we know that the effect of p^l is simply to add l to the power of p appearing in q . We satisfy ourselves with this observation on the powers of p when p is odd, and just remark that for even l this operation leaves the Weil index invariant. In particular, if $l < v_p(q)$ then multiplying by p^l or dividing by p^l yields the same Weil index. For $p = 2$, however, we can say more: Since the dependence on q in the expression for the Weil index of the 2-adic Jordan components in Proposition 4.1 is only via the power of ε , we find that multiplying the bilinear form by 2^l multiplies the Weil index by the l th power of the total sign of M (i.e., the product of the signs of all the Jordan components). In particular this term is just a sign, and the assertion about even l and division versus multiplication holds equally here (note that for odd p , or more accurately for $p \equiv 3 \pmod{4}$, multiplication by p does not necessarily change the Weil index only by a sign).

We turn now to describe the action of elements from \mathbb{Z}_p^* in the following

Lemma 4.2. *Let M be an even p -adic lattice and let $a \in \mathbb{Z}_p^*$. Then $\gamma(M(a))$ equals $(\frac{a}{\Delta_M})\gamma(M)$ for odd p and equals $(\frac{\Delta_M}{a})\gamma(M)^a$ (which is well-defined since the exponent is essentially $a \in \mathbb{Z}_2/8\mathbb{Z}_2 = \mathbb{Z}/8\mathbb{Z}$) for $p = 2$.*

Proof. Since both sides are multiplicative with respect to orthogonal direct sums, it suffices to verify the assertion for the Jordan components. Hence assume that M is a Jordan component of rank n , and write $M = M_e(q)$ with M_e unimodular of rank n . Then $\Delta_M = q^n$, and $M(a)$ is $(M_e(a))(q)$ with $M_e(a)$ unimodular of rank n (as $a \in \mathbb{Z}_p^*$). Hence we examine the effect of multiplication by a on the sign, and for $p = 2$ also on the index. Now, the discriminant is multiplied by a^n , so that the sign is multiplied by $(\frac{a}{p})^n$ for odd p and by $(\frac{2}{a})^n$ for $p = 2$. Moreover, for $p = 2$ we find that if M_e is odd then in any of its diagonal forms, $M_e(a)$ has all the diagonal elements multiplied by a , so that the index is multiplied by a . On the other hand, multiplying an even lattice by any term leaves it even, so that an index II remains unaffected. Altogether we find that if p is odd then $q^{\varepsilon n}(a)$ is isomorphic to $q^{(\frac{a}{p})^n \varepsilon n}$, while for $p = 2$ we see that $q_{t/II}^{\varepsilon n}(a)$ is isomorphic to $q_{at/II}^{(\frac{2}{a})^n \varepsilon n}$. When we substitute the values from Proposition 4.1 we note that $(\frac{a}{p})^{v_p(q)} = (\frac{a}{q})$ for odd p and $(\frac{2}{a})^{v_2(q)} = (\frac{q}{a})$ for $p = 2$ give the Legendre symbol with Δ_M . Furthermore, for $p = 2$, the $\varepsilon^{v_2(q)}$ part remains the same when raised to any odd power. Finally, the equality $\zeta_8^{at} = (\zeta_8^t)^a$ completes the case of odd M_e , while the fact that an index II is unaffected by multiplication by a and so does ζ_8^t with $t = 0$ verifies the assertion also for the even M_e case. This proves the lemma. \square

In addition to Lemma 4.2, we note that multiplying the bilinear form by -1 takes the Weil index to its complex conjugate, as is asserted in [W] and can be verified using Proposition 4.1. Moreover, substituting $a = -1$ in Lemma 4.2 and using the last assertion implies that $\overline{\gamma(M)}^2 = (\frac{-1}{\Delta_{M_p}})$ for odd p , and the

same applies for $\gamma(M)^2$ since this number is real. This observation can also be deduced directly from the formulae of Proposition 4.1 by multiplicativity. Lemma 4.2 and the last two assertions are useful when one wishes to compare the results of this paper with those of [Sche] and [Str]. Analogs of Lemma 4.2 for some special cases appear in Propositions 3.2 and 3.4 of [Sche].

In the following we shall need a more general Gauss sum, arising from a p -adic even lattice and a matrix in $SL_2(\mathbb{Z}_p)$. We define, for a p -adic even lattice M and a non-zero p -adic integer c , the subgroup $D_{M,c}$ of D_M to be the kernel of multiplication by c , and we denote its cardinality by $\Delta_{M,c}$. Moreover, we choose a Jordan decomposition of M , and define a vector $x_c \in M^*$ as follows. If $p \neq 2$ then we take $x_c = 0$. If $p = 2$ then we consider the lattice $M_{v_2(c)}$: If it comes with the index II then we take again $x_c = 0$. Otherwise, we take an orthogonal \mathbb{Z}_2 -basis for it and define x_c to be the half the sum of these basis vectors. Finally, we assume that a and c are relatively prime in \mathbb{Z}_p (i.e., not both are divisible by p) and define a_p to be $a/p^{v_p(a)}$ as above. Then we obtain

Theorem 4.3. *The Gauss sum $\sum_{\eta \in M/cM} \chi_p\left(\frac{a}{c}\frac{\eta^2}{2} + a\frac{x_c\eta}{c}\right)$ is well-defined and equals $p^{rk(M)v_p(c)/2} \sqrt{\Delta_{M,c}} \delta$ where $\delta = \prod_{pq|c} \gamma(q_{t/II}^{\varepsilon_n}(a_p c))$ and an empty product is defined (as always) to be 1.*

The index $/t/II$ appearing in δ means no index for odd p and means t or II according to what appears in that component for $p = 2$.

Proof. In order to see that the Gauss sum is well-defined, we note that by changing η by $c\lambda$ for some $\lambda \in M$ the argument of χ_p changes only by an element of \mathbb{Z}_p . Hence indeed each summand is well-defined. Moreover, since all the expressions involved (including $\Delta_{M,c}$ and the power of p) are multiplicative with respect to orthogonal direct sums, we just need to verify the assertions for a Jordan component. Furthermore, for any Jordan component with q and n the number $\Delta_{M,c}$ equals q^n if $v_p(q) \leq v_p(c)$ and equals $p^{v_p(c)n}$ if $v_p(q) \geq v_p(c)$. Finally, it is more convenient to multiply η by c_p (this is possible since $c_p \in \mathbb{Z}_p^*$), so that the summand corresponding to η is $\chi_p\left(ac_p \frac{\eta^2}{2p^{v_p(c)}} + a\frac{(\eta, x_c)}{p^{v_p(c)}}\right)$. We also recall that $q|\eta^2$ for every such η .

We distinguish among three different cases. The first case is where $p = 2$, $v_2(q) = v_2(c)$, and we have an index t ; this is the case where x_c is non-zero. The second case occurs whenever $v_p(q) \geq v_p(c)$ but excluding the situation covered in the first case (and then $x_c = 0$). Finally, the third case is the case where $v_p(q) < v_p(c)$, or equivalently $pq|c$ (and again $x_c = 0$). The proof in the second case is simple, since it is clear that the argument of χ_p is in \mathbb{Z}_p for every η (recall that for $p = 2$ and $v_p(c) = v_p(q)$ we assume that $\frac{\eta^2}{q}$ is even in this case), hence the total sum is $p^{nv_p(c)}$. Since $\Delta_{M,c} = p^{nv_p(c)/2}$ and $\delta = 1$ in this case, we obtain the desired result. We remark that this case covers the possibility where $a = 0$ and a_p is not defined (and more generally, the case where $p|a$), since then $c \in \mathbb{Z}_p^*$, the sum equals 1, and the product in δ is empty since no a with $pq|c$ exists (hence the value of a_p is irrelevant).

To prove the third case, note that if we change η by $\frac{p^{v_p(c)}}{q}\lambda$ with $\lambda \in M$ then we multiply the summand by $\chi_p(ac_p \frac{(\eta, \lambda)}{q})\chi_p(ac_p \frac{\lambda^2}{2q} \frac{p^{v_p(c)}}{q})$, and both arguments are in \mathbb{Z}_p (for the latter, note that since $p \mid \frac{p^{v_p(c)}}{q}$, it covers for the 2 in the denominator when $p = 2$). Therefore we can take out a multiplier of q^n and write the sum over $M/\frac{p^{v_p(c)}}{q}M$. Moreover, this case can occur only if $p \mid c$, so that we can replace a by a_p since p does not divide a . Now, as the image of the pairing of any two elements in $M = q^{\varepsilon_n}_{/t/II}$ is divisible by q and $q \mid p^{v_p(c)}$, we find that multiplying the bilinear form in M by $\frac{p^{v_p(c)}}{q^2}$ still gives a lattice. We denote this lattice by L , and its symbol is $(\frac{p^{v_p(c)}}{q})^{\varepsilon_n}_{/t/II}$. We claim that the sum in question, $\sum_{\eta \in M/\frac{p^{v_p(c)}}{q}M} \chi_p(a_p c_p \frac{\eta^2}{2p^{v_p(c)}})$, equals $\sum_{\rho \in L^*/L} \chi_p(a_p c_p \frac{\rho^2}{2})$. Indeed, L is even since $p \mid \frac{p^{v_p(c)}}{q}$, the dual lattice $L^* = \frac{q}{p^{v_p(c)}}L$ is M with the bilinear form divided by $p^{v_p(c)}$, and $L = \frac{p^{v_p(c)}}{q}L^*$. Combining these facts proves that the sums are equal. Since $\Delta_{L(a_p c_p)} = \Delta_L = \frac{p^{nv_p(c)}}{q^n}$, we deduce from Proposition 4.1 that the latter sum equals $\frac{p^{nv_p(c)/2}}{\sqrt{q^n}}\gamma(L(a_p c_p))$. Recall that the original sum was q^n times the latter, and note that $\Delta_{M,c} = q^n$ and $L(a_p c_p)$ has the same Weil index as $L(q^2 a_p c_p) = M(a_p c)$. The result for this case follows.

It remains to consider the first case, which can occur only if c and q are even (for the lattice to be even). Hence a is odd. As in the third case, replacing η by $\eta + 2\lambda$ multiplies $\chi_2(ac_2 \frac{\eta^2}{2^{v_2(c)+1}}) = \chi_2(ac_2 \frac{\eta^2}{2q})$ by $\chi_2(2ac_2 \frac{(\eta, \lambda)}{q})\chi_2(2ac_2 \frac{\lambda^2}{q})$ with the two arguments in \mathbb{Z}_2 . Moreover, writing $(\eta + 2\lambda, x_c)$ as $(\eta, x_c) + (\lambda, 2x_c)$ and recalling that $2x_c \in M$ shows that the argument of the multiplier $\chi_2(a \frac{(\lambda, 2x_c)}{q})$ by which $\chi_2(a \frac{(\eta, x_c)}{q})$ is changed is also in \mathbb{Z}_2 . Therefore in the total sum we can take a coefficient of $2^{n(v_2(c)-1)}$ out and carry out the summation on $M/2M$. Now, the fact that a and c_2 are odd shows that if we η as a linear combination an orthogonal \mathbb{Z}_2 -basis of M then $\chi_2(ac_2 \frac{\eta^2}{2q})$ is -1 raised to the power which is the sum of the coefficients. Indeed, an even coefficient contributes a 2-adic integer to the argument while an odd coefficient contributes a 2-adic half-integer. On the other hand, $\chi_2(a \frac{(\eta, x_c)}{q})$ is evaluated by the same argument as we used to evaluate $\chi_2(ac_2 \frac{\eta^2}{2q})$. Therefore the product of these two elements equals 1 for every η in $M/2M$. This completes the evaluation of the Gauss sum to $2^{nv_2(c)}$, and the same considerations as in the second case show that this is the value we need.

This proves the theorem. \square

We remark about the role of the element x_c here: In the case where it appears we have seen that the part with η^2 defines a character on $M/2M$, and x_c comes to cover for this character. This also shows that it is unique in some sense in D_M , since characters on M correspond bijectively to elements of M^* (as Section 1 shows), while with a different choice of character the Gauss sum

vanishes. This assertion will become more precise in Section 7, where we shall also see how x_c arises naturally in the evaluation of the action of SL_2 matrices via the operator \mathbf{r}_0 of [W]. We note that Theorem 3.9 of [Sche] (and some of its special cases which also appear in Section 3 of [Sche]) are related to our Theorem 4.3, although they evaluate other Gauss sums.

5 Metaplectic Groups over Local Fields

In this Section we explicitly construct the metaplectic cover of $SL_2(\mathbb{F})$ for a local field $\mathbb{F} \neq \mathbb{C}$ of characteristic $\neq 2$ as acting on \mathbb{F} -lattices, in terms of the Weil indices of the quadratic form on the lattice.

5.1 Hilbert Symbols and Weil Indices

Let \mathbb{F} be a local field of characteristic different from 2 which is not \mathbb{C} . Equivalently, either $\mathbb{F} = \mathbb{R}$, \mathbb{F} is a finite extension of \mathbb{Q}_p for some prime p , or \mathbb{F} is isomorphic to the field $\mathbb{F}_q((X))$ of formal Laurent series in one variable over the finite field \mathbb{F}_q of q elements for $q = p^e$ with some $e \geq 1$ and $p \neq 2$. We recall that the *Hilbert symbol* $(a, b)_{\mathbb{F}}$ is defined, for two elements a and b in \mathbb{F}^* , to be 1 if there exists a non-trivial solution to the equation $x^2 - ay^2 - bz^2$ in \mathbb{F} , and to be -1 if no such solution exists. The statement $(a, b)_{\mathbb{F}} = 1$ is equivalent to b being a norm from $\mathbb{F}(\sqrt{a})$ to \mathbb{F} of some element in $\mathbb{F}(\sqrt{a})$ (hence the Hilbert symbol is the norm residue symbol of exponent 2), or to the equation $x^2 - ay^2 - bz^2 + abt^2$ having some non-zero solution. As mentioned in Section 27 of [W], there are two isomorphism classes of non-degenerate quadratic forms in 4 variables over \mathbb{F} having a discriminant in $(\mathbb{F}^*)^2$, namely the reduced norm of the (unique) quaternion algebra over \mathbb{F} , and the “trivial” $xy + zt$. They can be differentiated by the latter representing 0 while the former fails to do so. Proposition 4 of [W] states that their Weil indices are different, as the former has -1 and the latter has 1 as their Weil indices (these statements hold also for characteristic 2). Since multiplying a quadratic form in an even number of variables by a non-zero constant neither changes its discriminant (up to $(\mathbb{F}^*)^2$) nor affects the question whether it represents 0 or not, we arrive at the following equation, which turns out to be very useful for our purposes:

$$\gamma(ux^2 - uay^2 - ubz^2 + uabt^2) = (a, b)_{\mathbb{F}},$$

where u , a , and b are elements in \mathbb{F}^* and x , y , z , and t are the variables of the quadratic form (see also Theorem A.4 in [Ra]). This is the reason why even though the Weil index of a quadratic form depends, in general, on the character one chooses on the field \mathbb{F} , the assertion in Proposition 4 of [W] (or equivalently the equation above) about the Weil index of a quadratic form does not involve the choice of the character: This is so because changing the character is equivalent to multiplying all the quadratic forms by some non-zero constant, and we have just seen that in this particular case such multiplication does not change the Weil index. Using the multiplicativity of the Weil indices with

respect to orthogonal direct sums, we obtain from the last equation that for any non-degenerate quadratic form q in m variables over \mathbb{F} (with any choice of the character λ on its additive group) the following relation holds:

$$\gamma(q)\overline{\gamma(aq)\gamma(bq)}\gamma(abq) = (a, b)_{\mathbb{F}}^m. \quad (2)$$

Since the right hand side in Eq. (2) is real, taking the complex conjugate of all the Weil indices in its left hand side gives the same result. Eq. (2) (and its complex conjugate) will turn out to be crucial for the remainder of this Section.

We remark that for $\mathbb{F} = \mathbb{C}$ all these statements hold trivially since all the Weil indices equal 1 and every quadratic form in more than one variable represents 0. The characteristic 2 case is excluded since the reduced norm of the quaternion algebra there looks different, and the corresponding theory of Hilbert symbols is more complicated. However, for the main result of this paper we use only the completions \mathbb{R} and \mathbb{Q}_p of \mathbb{Q} , which are of characteristic 0. We also remark that applying Eq. (2) with $\mathbb{F} = \mathbb{R}$ and using Milgram's identity for the corresponding discriminant forms yields a more precise version of Eq. (2.12) of [BS].

5.2 Unitary Operators on \mathbb{F} -Lattices

Choose a non-trivial character λ on the locally compact Abelian group $(\mathbb{F}, +)$. As already mentioned, any other choice is λ composed with multiplication by an element of \mathbb{F}^* . Let V be an \mathbb{F} -lattice of rank(=dimension) m . Since \mathbb{F} is a field, V is identified with its dual via the bilinear form. Let us denote this isomorphism by $\psi : V \xrightarrow{\sim} V^*$, with the convention that it acts from the right as in [W]. ψ is a symmetric morphism since the bilinear form is symmetric. We have an isomorphism $\lambda_* : V^* \xrightarrow{\sim} \widehat{V}$ (from the dual as a vector space to the Pontryagin dual) defined by composition with λ , i.e., $\lambda_*\varphi = \lambda \circ \varphi$. Then $\lambda_*\psi : V \xrightarrow{\sim} \widehat{V}$ is a symmetric isomorphism in the terminology of [W], and we normalize the Haar measure on V such that it equals the dual measure under this isomorphism. Since \mathbb{F} is not of characteristic 2, the pairing on V corresponds to the (unique) quadratic form $q : x \mapsto \frac{x^2}{2} = (x\psi)(\frac{x}{2})$. Then $f = \lambda \circ q$ is a (quadratic) non-degenerate character of second degree in the terminology of [W], which is associated to $\lambda_*\psi$. This setting (in characteristic 0, at least) corresponds closely to Example 2.19 of [Ge].

We now apply the process presented in Section 1 (where it was applied to D_M) to $G = V$. We restrict our attention to the subgroup $Sp_{\mathbb{F}}(V)$ of $Sp(V)$ consisting of those elements which preserve the \mathbb{F} -linear structure as well. Note that while in the most classical cases \mathbb{R} and \mathbb{Q}_p the linearity follows from additivity and continuity (since \mathbb{Q} is dense inside these fields), in general the structure of a locally compact Abelian group gives us also elements in $Sp(V)$ of [W] in which the “coordinates” are not \mathbb{F} -linear. Here the space $V \times V$ becomes a symplectic \mathbb{F} -vector space with the anti-symmetrization of the bilinear map on V . Hence the symplectic group $Sp_{\mathbb{F}}(V \times V)$ is defined. Identifying V with \widehat{V} via $\lambda_*\psi$ defines a natural isomorphism between the groups $Sp_{\mathbb{F}}(V)$ and $Sp_{\mathbb{F}}(V \times V)$. This map is pretty much the map μ defined in Section 33 of [W], and this process

is also presented following Example 2.21 of [Ge]. Since $2 \in \mathbb{F}^*$ we find that the group $Ps(V)$ of [W] is isomorphic to $Sp(V)$, and the corresponding “linear” subgroup $Ps_{\mathbb{F}}(V)$ is mapped onto $Sp_{\mathbb{F}}(V)$ under this isomorphism. Hence we identify all three groups $Sp_{\mathbb{F}}(V \times V)$, $Sp_{\mathbb{F}}(V)$, and $Ps_{\mathbb{F}}(V)$. Consider now the group of unitary operators on the space $L^2(V)$ (or on the dense subspace $\mathcal{S}(V)$ of Schwartz functions on V) which is denoted $Mp(V)$ in Section 34 of [W]. It is an S^1 -cover of $Ps(V)$. We denote $Mp_{\mathbb{F}}^{S^1}(V)$ the group of elements of $Mp(V)$ lying over $Ps_{\mathbb{F}}(V)$. By our identifications we consider it as an S^1 -cover of $Sp_{\mathbb{F}}(V \times V)$, and we denote the projection map by π .

Take a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $Sp_{\mathbb{F}}(V \times V)$, i.e., a, b, c , and d are elements in $End_{\mathbb{F}}(V)$ (acting from the right, as in [W]) which satisfy the symplectic condition with respect to the anti-symmetrization of the bilinear form on V . [W] provides formulae for the lift of such elements into $Mp_{\mathbb{F}}^{S^1}(V)$ in some cases, namely Eq. (16) there for invertible c and the appropriate combination $\mathbf{t}_0(f)\mathbf{d}_0(\alpha)$ (with f the quadratic one) for $c = 0$. In fact, the same formulae apply for elements of the groups without the \mathbb{F} index, but we prefer to restrict the discussion to $Mp_{\mathbb{F}}^{S^1}(V)$. Considering that we work only with $Ps(V)$ (and even $Ps_{\mathbb{F}}(V)$), we see that a change of variables and the symplectic conditions take Eq. (16) of [W] to a form in which Φ appears with a simple argument and the b (or β) coordinate does not appear at all. Under the isomorphism described in the previous paragraph and recalling the normalization of the Haar measure on V we find that if $c = 0$ then

$$\mathbf{r}_0(A)\Phi(x) = \sqrt{|\det a|_{\mathbb{F}}} \Phi(xa) \lambda \left[\frac{(xa, xb)}{2} \right],$$

and if c is invertible then

$$\mathbf{r}_0(A)\Phi(x) = \frac{1}{\sqrt{|\det c|_{\mathbb{F}}}} \int_V \Phi(y) \lambda \left[\frac{(yc^{-1}d, y)}{2} - (yc^{-1}, x) + \frac{(xac^{-1}, x)}{2} \right] dy,$$

where $|\cdot|_{\mathbb{F}}$ is the normalized absolute value of \mathbb{F} (i.e., $|\cdot|_{\mathbb{R}}$ is the usual absolute value, and if \mathbb{F} is non-archimedean with valuation v and residue field of cardinality q then $|u|_{\mathbb{F}} = q^{-v(u)}$ for $u \in \mathbb{F}^*$). In fact, [Ra] gives a formula for lifting more general elements of $Ps_{\mathbb{F}}(V)$ into $Mp_{\mathbb{F}}^{S^1}(V)$ (see part (3) of Lemma 3.2 there), but for our purposes the formulae from [W] will suffice. Theorem 4.1 of [Ra] implies that multiplying two elements of the form $\mathbf{r}_0(A)$ where at least one has zero c -entry gives the \mathbf{r}_0 of the product. Indeed, one can show that in this case two of the spaces appearing in the corresponding Leray invariant coincide, making the Leray invariant trivial. As Theorem 3 of [W] states, if all 3 c -entries are invertible then the product of the \mathbf{r}_0 elements gives the \mathbf{r}_0 of the product multiplied by the Weil index of a character of second degree which corresponds to the symmetric map given explicitly there. Since we are interested only in the quadratic characters of second degree over a field of characteristic $\neq 2$, this character of second degree is determined uniquely by the symmetric map. In relation with Theorem 4.1 of [Ra], one can also verify that the Leray invariant

indeed gives the asserted quadratic form (or character of second degree) in this case.

5.3 Elements of the Double Cover of $Sp_{\mathbb{F}}(V \times V)$

[W] shows that for any lattice V over a local field \mathbb{F} the S^1 -cover of $P_{S_{\mathbb{F}}}(V)$ contains a subgroup which is just a double cover of $P_{S_{\mathbb{F}}}(V)$. We denote this subgroup by $Mp_{\mathbb{F}}(V)$. It is determined by the S^1 -coefficients one takes before the elements in the set denoted $\Omega(G)$ (with $G = V$) in [W], i.e., those elements in which the γ coordinate (or, in our notation using the isomorphism arising from ψ , the c coordinate) is invertible. This set is Ω_n , with $n = \dim V$, in the notation of [Ra]. We note that under our isomorphism, the γ coordinate of the matrix corresponding to A from the previous section is just $\psi^{-1}c$.

We wish to determine the coefficients appearing in the inverse image of a certain subset of $Sp_{\mathbb{F}}(V \times V)$, using the Weil indices of our original quadratic form and its variants. This subset will suffice for our main goal, which is obtaining the representation of the double cover of $SL_2(\mathbb{F})$. The determination of the double cover has been done in a symplectic setting for all of $Sp_{\mathbb{F}}(V \times V)$ in Section 5 of [Ra], but it depends on the choice of a basis for V , which makes it not canonical. Our additional structure on V gives us a more canonical way to do this, at least for the subgroup $SL_2(\mathbb{F})$ which interests us. It may be possible to combine our method with that of [Ra] to obtain canonical formulae for larger subgroups of $Mp_{\mathbb{F}}(V)$ (see also the remark at the end of Section 9), but this is not necessary for the purpose of this paper. Unlike the formula in [Ra], we obtain our results directly from the assertions in [W].

Let c be an invertible element of $End_{\mathbb{F}}(V)$ such that $\psi^{-1}c : V^* \rightarrow V$ is symmetric. Then there exists a unique quadratic form on V which is associated to the inverse map $c^{-1}\psi$ (which is also symmetric). Call this map q_c , and its composition with λ gives a (quadratic) character of second degree f_c on V . Recall that both depend on ψ and the latter also depends on λ , but since we treat both ψ and λ as fixed from the outset (in particular ψ arises from the lattice structure on V), we omit them from the notation. Note that a map from V^* to V is symmetric in the sense of vector spaces if and only if the map from \widehat{V} to V obtained by composing with λ_* is symmetric in the sense of locally compact Abelian groups, so it is independent of the choice of λ . We now have the following

Lemma 5.1. *For any $A \in Sp_{\mathbb{F}}(V \times V)$ as above with invertible c -entry such that $\psi^{-1}c$ is symmetric, the intersection $\pi^{-1}(A) \cap Mp_{\mathbb{F}}(V)$ consists of the elements $\pm \gamma(f_c) \mathbf{r}_0(M)$. For such A with $c = 0$ and such that $\psi^{-1}a$ is symmetric (and evidently also invertible), this intersection consists of $\pm \gamma(f) \gamma(f_a) \mathbf{r}_0(A)$.*

Proof. We recall from Section 43 of [W] that $Mp_{\mathbb{F}}(V)$ is defined as the kernel of a certain character on $Mp_{\mathbb{F}}^{S^1}(V)$. This character takes an element of S^1 to its square and takes the element $\mathbf{r}_0(A)$ with A having invertible c -entry to $(D, -1)_{\mathbb{F}} \gamma(x^2)^{2m}$, where $m = \dim V$ and D is the determinant of $\psi^{-1}c$ using a basis for V and the dual basis of V^* . This expression is well-defined since

the Hilbert symbol is invariant under multiplying the entries by squares, and replacing the basis while keeping the duality condition changes D only by a square. In fact [W] distinguishes the cases of whether $-1 \in \mathbb{F}^*$ or not, where the formula just described corresponds to the latter case. In the former case the character is defined to attain 1 on such $\mathbf{r}_0(A)$. However, if $-1 \in \mathbb{F}^*$ then $(D, -1)_{\mathbb{F}} = 1$ for any D and $\gamma(x^2) = \pm 1$. Hence the formula given for the image of the character on the $\mathbf{r}_0(A)$ in the case $-1 \notin \mathbb{F}^*$ yields 1 if $-1 \notin \mathbb{F}^*$. Therefore we can use this formula for both cases. We have already taken into account the isomorphism from $Sp_{\mathbb{F}}(V \times V)$ to $Ps(V)$ when we replaced γ by $\psi^{-1}c$. This means that the coefficient t appearing in front of $\mathbf{r}_0(A)$ in an element of $Mp(V)$ has to satisfy

$$t^2 = \overline{(D, -1)_{\mathbb{F}} \gamma(x^2)^{2m}}.$$

Now, the inverse map $c^{-1}\psi$ has the same D as $\psi^{-1}c$ (up to squares, as usual). The quadratic form $2q_c$ also has the discriminant D , as is seen by choosing any basis for V which is orthogonal with respect to it. Hence q_c has the discriminant $2^{-m}D$. Then Eq. (28) of [W] yields $\gamma(f_c)^2 = (2^{-m}D, -1)_{\mathbb{F}} \gamma(x^2)^{2m}$. As the Hilbert symbol is bilinear and $(2, -1)_{\mathbb{F}} = 1$ (the triple $(1, 1, 1)$ is a solution to $x^2 - 2y^2 + z^2 = 0$), we can omit the factor 2^{-m} from the latter expression. This implies that $\gamma(f_c)$ has the desired square, and proves the first assertion.

To show the second assertion, we write $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -d \\ a & b \end{pmatrix}$, where both elements are symplectic. The right multiplier satisfies the condition of the first assertion by assumption, and the left multiplier satisfies it since ψ , hence also $-\psi^{-1}$, are symmetric. The corresponding characters of second degree are f_a and $-f$, and by the first assertion we can lift the multipliers to the \mathbf{r}_0 elements multiplied by $\pm \gamma(f_a)$ and $\pm \gamma(-f) = \gamma(f)$ respectively. As $\mathbf{r}_0(A)$ is the product of the \mathbf{r}_0 of the multipliers (as its c -entry vanishes), this proves the lemma. \square

We remark that the first assertion in Lemma 5.1 is in fact independent of ψ , since it is based on the (symmetric) isomorphism $\gamma = \psi^{-1}c$. On the other hand, the second assertion there does depend on the choice of ψ . We also note that by writing $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ as $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & d \\ -a & -b \end{pmatrix}$ we would obtain the complex conjugate coefficient $\pm \gamma(f) \gamma(f_a)$ instead. However, the square of this number is some Hilbert symbol multiplied by $\gamma(x^2)^{4m}$, and the latter is also ± 1 (as the Weil index of any quadratic form over a local field is an 8th root of unity). Hence this coefficient is a power of i and equals a sign times its complex conjugate. The latter assertion also follows from part (4) in Corollary A.5 of [Ra] in the 1-dimensional case, and using an orthogonal basis of V in the general case. The assertion of Lemma 5.1 holds for $\mathbb{F} = \mathbb{C}$ as well, yielding trivial results.

5.4 Representing $Mp_2(\mathbb{F})$ on \mathbb{F} -Lattices

For any local field \mathbb{F} [Ku1] constructed a double cover of $SL_2(\mathbb{F})$ which is non-trivial for $\mathbb{F} \neq \mathbb{C}$. In fact, this reference provides a construction of non-trivial covers of $SL_2(\mathbb{F})$ of any finite order, but we are interested here only in the double cover. Denote this double cover by $Mp_2(\mathbb{F})$. An element of $Mp_2(\mathbb{F})$ can be realized by a pair (A, ε) with $A \in SL_2(\mathbb{F})$ and $\varepsilon \in \{\pm 1\}$, and the product is defined by

$$(A, \varepsilon)(B, \delta) = (AB, \sigma(A, B)\varepsilon\delta)$$

where $\sigma(A, B)$ is the cocycle denoted in [Ku1] by $a(\sigma, \tau)$. The formula is

$$\sigma(A, B) = (x(A), x(B))_{\mathbb{F}}(x(AB), -x(B)/x(A))_{\mathbb{F}}$$

in the notation of [Ku1].

Let us write this cocycle explicitly for all the relevant cases. The symbol $x(A)$ (or $x(\sigma)$ in the notation of [Ku1]) depends on whether the c -entry of A is zero or not, which gives rise to 5 cases. In our case of the double cover the Hilbert symbol is defined on pairs of elements of $\mathbb{F}^*/(\mathbb{F}^*)^2$, and we can replace any division by multiplication (hence write $-x(A)x(B)$ instead of $-x(B)/x(A)$ in the equation of [Ku1]). In the case where $c = 0$ we can take either a or d as $x(A)$ since one is the inverse of the other. Note that $(u, -u)_{\mathbb{F}} = 1$ for any $u \in \mathbb{F}^*$ since $(0, 1, 1)$ is a solution to $x^2 - uy^2 + uz^2 = 0$. This observation and the multiplicative and symmetric properties of the Hilbert symbol allow us to write $(-cg, -cg)_{\mathbb{F}}$ as $(c, g)_{\mathbb{F}}(-c, -g)_{\mathbb{F}}$. Matrix multiplication shows that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ then $AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$, and the SL_2 condition implies

$$(ae + bg)c = -g + (ce + dg)a, \quad (cf + dh)g = -c + (ce + dg)h, \quad (3a)$$

$$(ae + bg)d = e + (ce + dg)b, \quad (cf + dh)e = d + (ce + dg)f. \quad (3b)$$

Summarizing, we obtain the formula

$$\sigma(A, B) = \begin{cases} (d, h)_{\mathbb{F}} = (a, e)_{\mathbb{F}} & c = g = 0 \\ (d, g)_{\mathbb{F}} = (a, g)_{\mathbb{F}} & c = 0 \neq g \\ (c, h)_{\mathbb{F}} = (c, e)_{\mathbb{F}} & c \neq 0 = g \\ (-c, -g)_{\mathbb{F}} & c \neq 0, g \neq 0, ce + dg = 0 \\ (c, g)_{\mathbb{F}}(ce + dg, -cg)_{\mathbb{F}} & c \neq 0, g \neq 0, ce + dg \neq 0 \end{cases} \quad (4)$$

(compare with Lemma 4.1 of [Str]).

Let V be an \mathbb{F} -lattice of dimension m , with the corresponding quadratic form q and character of second degree f obtained by composing q with a prefixed non-trivial character λ on $(\mathbb{F}, +)$. Following Example 2.21 of [Ge] (which is given for characteristic 0 but can be equally applied for all characteristics not equal to 2) we embed $SL_2(\mathbb{F})$ into $Sp_{\mathbb{F}}(V \times V)$ hence into $Ps_{\mathbb{F}}(V)$, and we seek the formulae for the representation of $SL_2(\mathbb{F})$ or its cover $Mp_2(\mathbb{F})$ that is obtained in this way. We shall identify $SL_2(\mathbb{F})$ with its images in $Sp_{\mathbb{F}}(V \times V)$

and in $Ps_{\mathbb{F}}(V)$ in what follows. By a slight abuse of notation we denote, for any $x \in \mathbb{F}^*$, the character of second degree $\lambda \circ (xq)$ (where q , hence the bilinear form on V , is multiplied by x) simply by xf . Then we have

Theorem 5.2. *The map $\rho_{V/\mathbb{F}}$ which takes an element (A, ε) in $Mp_2(\mathbb{F})$ to the operator $\varepsilon^m \overline{\gamma(cf)} \mathbf{r}_0(A)$ if $c \neq 0$ and $\varepsilon^m \overline{\gamma(af)} \gamma(f) \mathbf{r}_0(A)$ if $c = 0$ is a representation of $Mp_2(\mathbb{F})$ by elements of $Mp(V)$, which is faithful for odd m and factors through a faithful representation of $SL_2(\mathbb{F})$ if m is even.*

Proof. Recall the symmetric isomorphism $\psi : V \xrightarrow{\sim} V^*$ arising from the bilinear form on V . The maps $\psi^{-1}c$ for $c \neq 0$ and $\psi^{-1}a$ for $c = 0$ are clearly invertible and symmetric as nonzero scalar multiples of ψ^{-1} , and then f_c and f_a are simply $c^{-1}f$ and $a^{-1}f$ respectively. Since multiplication by any $x \in \mathbb{F}^*$ on V takes the character of second degree $x^{-1}f$ to xf , we find that $\gamma(x^{-1}f) = \gamma(xf)$. Hence the fact that the representing elements lie in $Mp(V)$ follows from Lemma 5.1. Since the composition $\pi \circ \rho_{V/\mathbb{F}}$ is just the projection from $Mp_2(\mathbb{F})$ to $SL_2(\mathbb{F})$ we see that $\rho_{V/\mathbb{F}}$ is a “representation up to sign”. Moreover, the two elements in the same coset of ± 1 are sent via ρ to the same element if and only if m is even. This implies that once ρ is a true representation, it is faithful for odd m and faithful of $SL_2(\mathbb{F})$ for even m . It remains to show that ρ respects the sign in the product rule of $Mp_2(\mathbb{F})$. As the action of ε is just multiplication by ε^m , we need to check it only for the products $(A, 1)(B, 1)$.

Except for the case in which all of c , g , and $ce + dg$ are non-zero, the product of $\mathbf{r}_0(A)$ and $\mathbf{r}_0(B)$ is $\mathbf{r}_0(AB)$. Otherwise we have to multiply by the Weil index of the quadratic character of second degree corresponding to the symmetric map $(\psi^{-1}c)^{-1}(\psi^{-1}(ce + dg))(\psi^{-1}g)^{-1} = \frac{ce+dg}{cg}\psi$, which is simply $\gamma(\frac{ce+dg}{cg}f)$ (this can also be seen using the Leray invariant). Therefore, if η_A is the coefficient preceding $\mathbf{r}_0(A)$ in $\rho_{V/\mathbb{F}}(A)$ then we have to show that $\frac{\eta_A \eta_B \gamma}{\eta_{AB}} = \sigma(A, B)^m$ where γ is $\gamma(\frac{ce+dg}{cg}f)$ if c , g , and $ce + dg$ are all non-zero and 1 otherwise. Observe that multiplication of the quadratic form by a square gives an isomorphic quadratic form which has the same Weil index. Hence in the case where $c = 0$ we have $\gamma(af) = \gamma(df)$, and the non-trivial Weil index appearing as γ here can be replaced by $\gamma(cg(ce + dg)f) = \gamma(-cg(ce + dg)f)$. If $ce + dg = 0$ where c and g are non-zero then Eq. (3a) implies $cf + dh = -\frac{c}{g}$ and $ae + bg = -\frac{g}{c}$. Hence multiplying f by any of these numbers gives the Weil index $\gamma(cgf)$. Listing the roots of unity η_A , η_B , γ , and $\overline{\eta_{AB}}$ according to the various cases of Eq. (4) gives

$$\left\{ \begin{array}{llll} \frac{\gamma(af)\gamma(f)}{\gamma(df)\gamma(f)} & \frac{\gamma(ef)\gamma(f)}{\gamma(gf)} & 1 & \gamma(aef)\overline{\gamma(f)} \\ \gamma(cf) & \gamma(ef)\gamma(f) & 1 & \gamma(dgf) \\ \gamma(-cf) & \gamma(-gf) & 1 & \gamma(cef) \\ \gamma(cf) & \gamma(gf) & \overline{\gamma(-cg(ce + dg)f)} & \gamma(cgf)\gamma(f) \end{array} \right. \gamma((ce + dg)f).$$

Presented in this way it is clear that Eq. (2) and its complex conjugate imply that the product of the elements in any of the first four rows gives the corresponding case of $\sigma(A, B)^m$. The same conclusion holds for the last row once we

multiply the first two elements by $\gamma(f)\gamma(cgf)$ and the last two elements by its inverse $\gamma(f)\gamma(-cgf)$. This completes the proof of the Theorem. \square

Theorem 5.2 determines the structure of the subgroup of $Mp(V)$ which lies over $SL_2(\mathbb{F})$: If m is odd, then it is isomorphic to $Mp_2(\mathbb{F})$, while if m is even, then it is just a trivial double cover isomorphic to $SL_2(\mathbb{F}) \times \{\pm 1\}$. We note that Theorem 5.2 holds trivially also for $\mathbb{F} = \mathbb{C}$, with all the factors being 1, giving a representation of $SL_2(\mathbb{C})$ in any case, since $Mp_{\mathbb{C}}^{S^1}(V)$ already splits. We note that for $m = 1$, Theorem 5.2 agrees with the construction of Section 5 of [Ra]. Hence it implies the order 2 case of [Ku1], as mentioned in [Ra]. Then one can decompose V into 1-dimensional subspaces using an orthogonal basis, and deduce Theorem 5.2 from the results of [Ra]. However, Theorem 5.2 is not a special case of the results of [Ra], as can be seen by taking V to be an 8-dimensional vector space over \mathbb{F} with a quadratic form with Weil index not in $\{\pm 1\}$: Then for $c = 1$ the formula of [Ra] yields the coefficient 1, while Theorem 5.2 gives a coefficient not in $\{\pm 1\}$. The reason for this is probably the fact that different 1-dimensional quadratic forms correspond to different characters on $(\mathbb{F}, +)$ in the symplectic terminology, while [Ra] considers only one such character. This technical difficulty is the reason why it seems simpler to prove Theorem 5.2 directly, rather than to translate of the results of [Ra] to our setting.

We note at this point that for $\mathbb{F} = \mathbb{R}$ we already have the realization of $Mp_2(\mathbb{R})$ using the square root of the function $j(A, \tau)$ presented in Section 1. We recall that the Hilbert symbol $(a, b)_{\mathbb{R}}$ equals 1 if a or b are positive and -1 if both a and b are negative. One can verify that by sending $(A, 1)$ to the element $(A, \sqrt{j(A, \tau)})$ with the square root having argument in $[-\frac{\pi}{2}, \frac{\pi}{2})$, we obtain exactly the same cocycle as in Eq. (4), yielding the explicit isomorphism between the “abstract” and “modular” $Mp_2(\mathbb{R})$. Indeed, the only places where the product of elements in the “modular” $Mp_2(\mathbb{R})$ with this branch of the square root multiply to give an element with the other branch of the square root are found in the following cases (ordered according to the possibilities of Eq. (4)): $c = g = 0$ and negative d and h ; $c = 0$ and negative d and g ; $g = 0$ and negative c and h ; positive c and g with $ce + dg = 0$; and c and g having both a sign opposite of that of $ce + dg$ if none of these three numbers vanishes. Indeed, these are precisely the cases in which the cocycle in Eq. (4) over \mathbb{R} gives -1 . Therefore, composing $\rho_{V/\mathbb{R}}$ with this isomorphism gives the representation of the “modular” $Mp_2(\mathbb{R})$. For this case, Theorem 5.2 reproduces Lemma 1.3 of [Sh], and the isomorphism between the “abstract” and “modular” $Mp_2(\mathbb{R})$ appears implicitly also in this reference. We note that both [Sh] and [Str] use the alternative branch of $\sqrt{j(A, \tau)}$, with an argument in $(-\frac{\pi}{2}, \frac{\pi}{2}]$. One can see, using the same considerations, that the cocycle obtained using the branch is similar to the one from Eq. (4), but with c , g , and $ce + dg$ all multiplied by a minus sign. In fact, the bilinearity of the Hilbert symbol implies that this change does not affect the fifth case of Eq. (4), in correspondence with the fact that for elements with non-zero c -entry both choices of arguments yield the same element of $Mp_2(\mathbb{R})$. This observation shows up even more clearly when

one uses the form obtained in Lemma 6.2 below. The observation that the difference between this latter cocycle and the one of Eq. (4) is compensated by inverting the sign for elements with $c = 0$ and $d < 0$ is precisely the content of Theorem 4.2 of [Str]. Moreover, the fact that the cocycle in this representation is σ^m is related to Eq. (5.1) of [Str].

We also note that applying the process described in Section 1 for obtaining a representation of $Mp_2(\mathbb{Z})$ to this case gives the composition of the map ρ with the natural map from \mathbb{Z} to \mathbb{F} . This is easily verified by checking the actions of T and S . This procedure shows that Theorem 5.2 contains Theorem 2.22 of [Ge] as a special case, and the assertion about the factoring of $\rho_{V/\mathbb{F}}$ through $Mp_2(\mathbb{Z})$ if and only if m is even is equivalent to Corollary 2.24 of this reference.

6 Working over the Ring of Integers

In this Section we present, for a non-archimedean local field \mathbb{F} , a lift from a certain congruence subgroup of SL_2 over the ring of integers in \mathbb{F} (which is almost always the entire group), and relate the inverse images of $SL_2(\mathbb{Z})$ in $Mp_2(\mathbb{R})$ and $Mp_2(\mathbb{Q}_2)$.

6.1 Some Results on Hilbert Symbols

Let us assume that \mathbb{F} is non-archimedean (of characteristic $\neq 2$), with ring of integers \mathcal{O} , uniformizer π , and valuation v . We wish to present a lift of a certain subgroup of $SL_2(\mathbb{F})$ (or more specifically, of $SL_2(\mathcal{O})$) into $Mp_2(\mathbb{F})$. As a preliminary step, we evaluate some Hilbert symbols. First we extend the previous notation, and write any $x \in \mathcal{O}$ as $\pi^{v(x)}x_\pi$. We now claim that any element in $1 + 4\pi\mathcal{O}$ is a square in \mathcal{O}^* . Indeed, for any element a in that set, taking the polynomial $P(x) = x^2 - a$ with the value $x = 1$ gives that $P'(1) = 2$ has valuation $v(2)$ while $P(1) = 1 - a$ has valuation larger than $2v(2)$. Hence by Hensel's Lemma we can find a solution for $P(x) = 0$ which is congruent to 1 modulo $\pi\mathcal{O}$.

We now prove the following

Lemma 6.1. *Let $u \in \mathcal{O}^* \cap (1 + 4\mathcal{O})$, $v \in \mathcal{O}^*$, $y \in \mathcal{O}$, and $t \in \mathcal{O}$. Then the following assertions hold: (i) $(u, v)_{\mathbb{F}} = 1$. (ii) $(u, y)_{\mathbb{F}} = (u, \pi^{v(y)})_{\mathbb{F}}$. (iii) Assume $4|y$. Assume further that either $\pi|\frac{y}{4}$, $y = \pi^{v(y)}$, or $u + ty \in \mathcal{O}^*$ (the latter always happens if 2 is not in \mathcal{O}^*). Then we have $(u + ty, y)_{\mathbb{F}} = (u, y)_{\mathbb{F}}$.*

Part (i) of Lemma 6.1, specialized to the cases where the characteristic of the residue field of \mathbb{F} differs from 2 or equals 2, agrees with parts (ii) and (iii) of Proposition 2.1 of [Ge] respectively. We provide the proof of this part as well for the sake of completeness.

Proof. If the characteristic of the residue field is not 2 (so that the assumption $u \in 1 + 4\mathcal{O}$ is redundant) then $|\frac{(\mathcal{O}/\pi\mathcal{O})^2}{2}| = \frac{|\mathcal{O}/\pi\mathcal{O}|+1}{2}$ (including 0). Therefore the expressions $1 - uy^2$ and vz^2 with y and z in \mathcal{O} both have this number

of possible images modulo $\pi\mathcal{O}$. Put together, this gives rise to $|(\mathcal{O}/\pi\mathcal{O})| + 1$ values. Hence there exist y and z such that these expressions have the same value modulo $\pi\mathcal{O}$. With these y and z we have that $uy^2 + vz^2 \in 1 + 4\pi\mathcal{O}$ (as $4 \in \mathcal{O}^*$) and is a square, which yields a solution to $x^2 - uy^2 - vz^2 = 0$. On the other hand, if the characteristic of the residue field is 2 (hence $1 + 4\mathcal{O} \subseteq \mathcal{O}^*$) then every element in the residue field is a square in that field. In particular there exists some $s \in \mathcal{O}$ such that $s^2 \equiv \frac{1-u}{4v} \pmod{\pi\mathcal{O}}$ (recall that $v \in \mathcal{O}^*$). But then $u + 4s^2v \in 1 + 4\pi\mathcal{O}$ and is a square, yielding a solution of $x^2 - uy^2 - vz^2 = 0$. This establishes part (i). For part (ii) write $y = y_\pi \pi^{v(y)}$. Then the bilinearity of the Hilbert symbol and part (i) immediately imply the desired assertion. For part (iii) we note that if $\pi \mid \frac{y}{4}$ then the ratio $1 + \frac{t}{u}y$ is in $1 + 4\pi\mathcal{O}$ hence is a square. The assertion follows in this case from the bilinearity of the Hilbert symbol. Otherwise, part (ii) shows that under either hypothesis we have to compare $(u, \pi^{v(y)})$ and $(u + ty, \pi^{v(y)})$. But $\frac{y}{4}$ is in \mathcal{O}^* and $v(y) = v(4)$ is even. Therefore both Hilbert symbols involve an element in $(\mathbb{F}^*)^2$, hence both equal 1. This completes the proof of the lemma. \square

Since $(\pi, -\pi)_{\mathbb{F}} = 1$ it follows that $(\pi, \pi)_{\mathbb{F}} = (-1, \pi)_{\mathbb{F}}$. In fact, combining this assertion with Lemma 6.1 completely determines the Hilbert symbol over \mathbb{F} if the residue characteristic is not 2 (and $|\mathbb{F}^*/(\mathbb{F}^*)^2| = 4$ in this case). Moreover, in this case there can be only one square root of an element in $1 + 4\pi\mathcal{O} = 1 + \pi\mathcal{O}$ which is congruent to 1 modulo $\pi\mathcal{O}$. This is so since multiplying by -1 gives an element without this property (since $2 \notin \pi\mathcal{O}^*$ hence $-1 \not\equiv 1 \pmod{\pi\mathcal{O}}$). Thus, we have a “principal branch” of the square root on that set. We claim that such a “principal branch” exists also when 2 is not in \mathcal{O}^* . To see this, note that any root of an element in $1 + 4\pi\mathcal{O}$ (and even in $1 + 4\mathcal{O}$) has to be congruent to 1 modulo 2. This follows from the observation that if $0 < r < v(2)$ and $t \in \mathcal{O}^*$ then $v((1 + \pi^r t)^2 - 1) = 2r < v(4) + 1$. Then writing the root as $1 + 2t$ with $t \in \mathcal{O}$ then the equality $(1 + 2t)^2 = 1 + 4t + 4t^2$ implies that t has to satisfy $\pi \mid t^2 + t$. But this implies the image of t modulo $\pi\mathcal{O}$ is in the kernel of the $(\mathbb{F}_2$ -linear) Artin–Schreier map $s \mapsto s^2 + s$ on $\mathcal{O}/\pi\mathcal{O}$, so that t lies in the primary subfield \mathbb{F}_2 of $\mathcal{O}/\pi\mathcal{O}$. The identity $-(1 + 2t) = 1 - 2(t + 1)$ implies that for the other root we obtain the the other image in \mathbb{F}_2 . Thus, we have a “principal” square root of elements in $1 + 4\pi\mathcal{O}$ with image in $1 + 2\pi\mathcal{O}$. In fact, this “principal” square root (either if the characteristic of the residue field is 2 or not) is the root whose existence is obtained by Hensel’s Lemma.

We recall the well-known fact that if the characteristic of the residue field differs from 2 then $\mathcal{O}^*/(\mathcal{O}^*)^2$ is of order 2 (since only the group of units in the residue field modulo squares remains). Part (i) of Lemma 6.1 shows that an element $u \in \mathcal{O}^*$ satisfies $(u, \pi)_{\mathbb{F}} = 1$ if and only if u is a square, since the Hilbert symbol is non-degenerate. In particular, the last assertion for $\mathbb{F} = \mathbb{Q}_p$ implies that the extension of the usual Legendre symbol to $(\frac{x}{p^k})$ with $x \in \mathbb{Z}_p$ in Section 4 coincides, if x and p^k are coprime in \mathbb{Z}_p , with the Hilbert symbol $(x, p^k)_{\mathbb{Q}_p}$. Indeed, some papers including [Ge] and [Ku2] define a Legendre symbol for more general non-archimedean fields with residue characteristic $\neq 2$, and this Legendre symbol also coincides with the corresponding Hilbert symbol.

However, we use here only the case $\mathbb{F} = \mathbb{Q}_p$ of this assertion. We therefore use the Legendre and the Hilbert symbols interchangeably for this case in what follows. Considerations similar to those of the previous paragraph show that the same assertions hold in residue characteristic 2 for elements in $1 + 4\mathcal{O}$. Indeed, according to these arguments, $u \in 1 + 4\mathcal{O}$ is a square if and only if the image of $\frac{u-1}{4}$ modulo $\pi\mathcal{O}$ is in the image of the Artin–Schreier map. This map is \mathbb{F}_2 -linear and has a kernel of cardinality 2, hence exactly half of the quotient group $(1 + 4\mathcal{O})/(1 + 4\pi\mathcal{O}) \cong (\mathcal{O}/\pi\mathcal{O}, +)$ consists of squares. The assertion that $u \in 1 + 4\mathcal{O}$ satisfies $(u, \pi)_{\mathbb{F}} = 1$ if and only if u is a square follows from Lemma 6.1 and the non-degeneracy of the Hilbert symbol as in the case of odd residue characteristic.

We recall the expression appearing in the last case of Eq. (4). It is expedient to develop equivalent forms for this expression, and these are given in the following

Lemma 6.2. *Let c, g, d , and e be elements of \mathbb{F} such that c, g , and $x = ce + dg$ are non-zero (hence d and e cannot both be 0). Then $(c, g)_{\mathbb{F}}(ce + dg, -cg)_{\mathbb{F}}$ equals $(e, -cg)_{\mathbb{F}}$ if $d = 0$, $(d, -cg)_{\mathbb{F}}$ if $e = 0$, and $(d, cx)_{\mathbb{F}}(e, gx)_{\mathbb{F}}(d, e)_{\mathbb{F}}$ if both d and e are non-zero.*

Proof. $(c, -c)_{\mathbb{F}} = 1$ implies $(c, g)_{\mathbb{F}} = (c, -cg)_{\mathbb{F}}$, so that the expression from Eq. (4) becomes $(-cg, cx)_{\mathbb{F}} = (-cg, \frac{x}{c})_{\mathbb{F}}$. We now substitute $\frac{x}{c} = e + \frac{dg}{c}$. Now, if $d = 0$ then $\frac{x}{c} = e$, which proves the assertion in this case. In the case $e = 0$ we have $c^2 \frac{x}{c} = cdg$, and since $(-cg, cg)_{\mathbb{F}} = 1$ this case is also complete. We assume now that both d and e are non-zero, and write our expression as $(-cg, e)_{\mathbb{F}}(-cg, 1 + \frac{dg}{ce})_{\mathbb{F}}$. Now, the equality $(t, 1 - t)_{\mathbb{F}} = 1$ holds for every $t \neq 0, 1$ (as $(1, 1, 1)$ is a solution to the corresponding equation). Multiplying this Hilbert symbol with $t = -\frac{dg}{ce}$ (which does not equal 0 or 1 by our assumptions) by the second Hilbert symbol in our expression (and substituting x again) yields $(\frac{dg^2}{e}, \frac{x}{ce})_{\mathbb{F}} = (de, cex)_{\mathbb{F}}$. Inserting the first Hilbert symbol again gives $(d, cex)_{\mathbb{F}}(e, -c^2gex)_{\mathbb{F}}$, and since the latter Hilbert symbol equals $(e, gx)_{\mathbb{F}}$ (as $(e, -e)_{\mathbb{F}} = 1$), the bilinearity of the Hilbert symbol leads us to the desired conclusion. \square

6.2 Splitting over $\Gamma_1(4, \mathcal{O})$

Next, we present a lift of the group $\Gamma_1(4, \mathcal{O})$ of matrices in $SL_2(\mathcal{O})$ whose c -entry is divisible by 4 in \mathcal{O} and such that their a -entry, or equivalently their d -entry, is congruent to 1 modulo 4. Of course, if $2 \in \mathcal{O}^*$, i.e., if the characteristic of the residue field is not 2, then this group is just $SL_2(\mathcal{O})$. This result is already stated in Proposition 2.8 of [Ge] and proven as Theorem 2 of [Ku2]: Note that intersecting the group denoted K^N with $N = 4$ in [Ge] with $SL_2(\mathcal{O})$ gives precisely the group $\Gamma_1(4, \mathcal{O})$ with which we work here. We include the proof here, not just for the exposition to be self-contained, but also because the calculations in [Ku2] use the norm residue symbol of general rank and are extended to covers of $GL_2(\mathbb{F})$. This renders them much more complicated than those needed for our

particular case. Moreover, the assumptions of [Ge] and [Ku2] are stronger than ours (to name one difference, the characteristic 0 is assumed in both references, while we require only a characteristic $\neq 2$). The proof presented here differs from that of [Ku2], even though it is also computational. Moreover, with slight changes it can be applied for $GL_2(\mathbb{F})$ with the cocycle extending σ which appears in [Ku2] and [Ge]: One just has to divide (or equivalently for Hilbert and Legendre symbols, multiply) d by $\det A$, g and h by $\det B$, and c and x by $\det AB$, and add $\det A$ (resp. $\det B$) in the appropriate place in the left (resp. right) hand side in Eqs. (3a) and (3b). The expression involving v in [Ku2] and [Ge], which is $(d \det A, \det B)_{\mathbb{F}}$ for $c = 0$, equals 1 also in this case: Note that $\det A = ad$ and $a \in \mathcal{O}^* \cap (1 + 4\mathcal{O})$, and use part (i) of Lemma 6.1. With these changes, the statement and proof of Theorem 6.3 extend precisely to the present case by applying Lemma 6.2 with the appropriate numbers. In this paper, however, we restrict attention to $SL_2(\mathbb{F})$.

The result can be stated as

Theorem 6.3. *The map $\iota : \Gamma_1(4, \mathcal{O}) \rightarrow Mp_2(\mathbb{F})$ which takes the matrix A to $(A, 1)$ if $c = 0$ and to $(A, (a_{\pi}, \pi^{v(c)})_{\mathbb{F}}) = (A, (d_{\pi}, \pi^{v_{\pi}(c)})_{\mathbb{F}})$ if $c \neq 0$ is a lift which is a group homomorphism.*

Proof. The fact that ι is a lift (hence in particular injective) is immediate from the definition. We can interchange a and d with a_{π} and d_{π} , for the following reason: If $\pi|c$ (which always happens in residue characteristic 2) then both a and d are in \mathcal{O}^* and otherwise the Hilbert symbol with $\pi^0 = 1$ is just 1 in any case (we include in that manner also the case where $a = 0$ or $d = 0$, ignoring the fact that a_{π} and d_{π} are not well-defined then). Hence we use a and d in the proof but write a_{π} and d_{π} in the assertion. Moreover, as $4|\pi^{v(c)}$ and we can write $ad = 1 + (bc_{\pi})\pi^{v(c)}$ with $bc_{\pi} \in \mathcal{O}$, part (iii) of Lemma 6.1 yields $(ad, \pi^{v(c)})_{\mathbb{F}} = 1$ hence $(a, \pi^{v(c)})_{\mathbb{F}} = (d, \pi^{v(c)})_{\mathbb{F}}$. We now write $\iota(A) = (A, \varepsilon_A)$ and verify that ι is a group homomorphism. This amounts, using the definition of $Mp_2(\mathbb{F})$, to verifying that the difference in signs $\varepsilon_A \varepsilon_B \varepsilon_{AB}$ equals the value of the cocycle $\sigma(A, B)$ from Eq. (4) for matrices in $\Gamma_1(4, \mathcal{O})$. Since the ε terms and other Hilbert symbols are signs ± 1 , we write from now on quotients as products (and cancel Hilbert symbols in products as if they were quotients). Note that ε_A is just $s(k)$ of Lemma 2.9 of [Ge]. Indeed, for $c = 0$ it equals 1 by definition, and for $c \in \mathcal{O}^*$ it equals 1 since $\pi^{v(c)} = 1$. On the other hand, if $c \neq 0$ and $\pi|c$ then d (or, in the $GL_2(\mathbb{F})$ setting, $d \det A$) is in $\mathcal{O}^* \cap (1 + 4\mathcal{O})$. Then part (ii) of Lemma 6.1 implies $(d, \pi^{v(c)})_{\mathbb{F}} = (d, c)_{\mathbb{F}}$.

We consider each of the cases of Eq. (4) separately. In the first case we have $\varepsilon_A = \varepsilon_B = \varepsilon_{AB} = 1$. Since a and e (or equivalently d and h) are in $\mathcal{O}^* \cap (1 + 4\mathcal{O})$, part (i) of Lemma 6.1 shows that $\sigma(A, B) = 1$ as well. In the second case $\varepsilon_A = 1$, and since $v(dg) = v(g)$ we have $\varepsilon_B \varepsilon_{AB} = (d, \pi^{v(g)})_{\mathbb{F}}$ (also when $h = 0$ since then $\pi^{v(g)} = 1$). This equals $(d, g)_{\mathbb{F}}$ by part (ii) of Lemma 6.1. The third case is similar: We have $\varepsilon_B = 1$ and $\varepsilon_A \varepsilon_{AB} = (e, \pi^{v(c)})_{\mathbb{F}}$ since $v(ce) = v(c)$ (also when $a = 0$), yielding $(c, e)_{\mathbb{F}}$ by part (ii) of Lemma 6.1. In the fourth case $\varepsilon_{AB} = 1$ and we see, since $AB \in \Gamma_1(4, \mathcal{O})$, that $-\frac{g}{c} \in \mathcal{O}^* \cap (1 + 4\mathcal{O})$. Hence $v(c) = v(g)$ and we denote this valuation by v . Now, the equality $e = -\frac{dg}{c}$

implies $\varepsilon_A \varepsilon_B = \left(-\frac{g}{c}, \pi^v\right)_{\mathbb{F}}$ (this also holds if $d = e = 0$ since then $\pi^v = 1$), while $(-c, c)_{\mathbb{F}} = 1$ implies $(-c, -g)_{\mathbb{F}} = \left(-c, -\frac{g}{c}\right)_{\mathbb{F}}$. As $-\frac{g}{c} \in \mathcal{O}^* \cap (1 + 4\mathcal{O})$, part (ii) of Lemma 6.1 and the symmetry of the Hilbert symbol show that the two Hilbert symbols in question have the same value, as desired.

Finally, for the fifth case we use Lemma 6.2 and separate into cases according as to whether d and e are in \mathcal{O}^* or not. We begin by assuming that both d and e are in \mathcal{O}^* , hence in $\mathcal{O}^* \cap (1 + 4\mathcal{O})$. According to parts (i) and (ii) of Lemma 6.1, the expression in Lemma 6.2 is just

$$(d, \pi^{v(c)})_{\mathbb{F}} (d, \pi^{v(x)})_{\mathbb{F}} (e, \pi^{v_\pi(g)})_{\mathbb{F}} (e, \pi^{v(x)})_{\mathbb{F}}.$$

The first and third Hilbert symbols here are ε_A and ε_B respectively. Now, Eq. (3b) shows that ε_{AB} can be written as either one of the Hilbert symbols $(de + bdx, \pi^{v(x)})_{\mathbb{F}}$ and $(de + efx, \pi^{v(x)})_{\mathbb{F}}$. Since $4 \mid \pi^{v(x)}$, part (iii) of Lemma 6.1 that any of these Hilbert symbols is indeed the product of the second and fourth Hilbert symbols in the above expression. This completes the proof of the theorem for the case where the residue characteristic of \mathbb{F} is 2. Next, we assume that this characteristic differs from 2 (so that the condition $4 \mid y$ in part (iii) of Lemma 6.1 holds for any $y \in \mathcal{O}$), and consider the other cases. If d is in \mathcal{O}^* and e is not then $g \in \mathcal{O}^*$, $x \in \mathcal{O}^*$, $\varepsilon_B = \varepsilon_{AB} = 1$, and by part (ii) of Lemma 6.1 we have that $(d, cx)_{\mathbb{F}} = (d, -cg)_{\mathbb{F}} = \varepsilon_A$. In the case $e = 0$, Lemma 6.2 immediately gives the desired result. Otherwise, the remaining factor $(e, dgx)_{\mathbb{F}} = (e, x^2 - cex)_{\mathbb{F}}$ is also 1 by part (iii) of Lemma 6.1. Similarly, if $e \in \mathcal{O}^*$ and d is not then c and x are invertible, $\varepsilon_A = \varepsilon_{AB} = 1$, $(e, gx)_{\mathbb{F}} = (e, -cg)_{\mathbb{F}} = \varepsilon_B$ by part (ii) of Lemma 6.1, and for non-zero d we have $(d, cex)_{\mathbb{F}} = (d, x^2 - dgx)_{\mathbb{F}} = 1$ by part (iii) of Lemma 6.1. Finally, if d and e are both not invertible then we use again the original expression from Eq. (4). We then observe that since c and g are in \mathcal{O}^* we have $\varepsilon_A = \varepsilon_B = 1$, while part (i) of Lemma 6.1 implies $(c, g)_{\mathbb{F}} = 1$. Eq. (3a) now shows that $\varepsilon_{AB} = (-cg, \pi^{v(x)})_{\mathbb{F}}$, which by part (ii) of Lemma 6.1 yields the desired value $(-cg, x)_{\mathbb{F}}$ since $-cg \in \mathcal{O}^*$ (and is also in $1 + 4\mathcal{O}$).

The proof of the theorem is now complete. \square

For odd residue characteristic, $\Gamma_1(4, \mathcal{O}) = SL_2(\mathcal{O})$ and Theorems 5.2 and 6.3 combine to give the following

Corollary 6.4. *If the residue characteristic of \mathbb{F} is odd then for an \mathbb{F} -lattice V of dimension m , the map which takes a matrix $A \in SL_2(\mathcal{O})$ (with the usual entries) to $(a_\pi, \pi^{v(c)})_{\mathbb{F}}^m \overline{\gamma(cf)} \mathbf{r}_0(A)$ if $c \neq 0$ and to $\overline{\gamma(af)} \gamma(f) \mathbf{r}_0(A)$ if $c = 0$ is a faithful representation of $SL_2(\mathcal{O})$ on $L^2(V)$, with respect to which $S(V)$ is a (dense) invariant subspace.*

Proof. Indeed, this map is just the composition $\rho_{V/\mathbb{F}} \circ \iota$, which is a section of the restriction of π to $SL_2(\mathcal{O})$. \square

Remaining in the case of the odd residue characteristic, part (i) of Lemma 6.1 gives in particular $(-1, -1)_{\mathbb{F}} = 1$. Therefore the quadratic form $x^2 + y^2 + z^2 + w^2$ represents 0, hence all the Weil indices over \mathbb{F} are powers of i . This implies that

the process of Section 1 yields a representation of $SL_2(\mathbb{Z})$ (avoiding the double cover $Mp_2(\mathbb{Z})$). Indeed, this representation is the one obtained from Corollary 6.4 via the natural map from \mathbb{Z} to \mathcal{O} , as one can verify by checking the images of T and S .

6.3 Normality of the Lift

If the residue characteristic of \mathbb{F} is 2 (i.e., \mathbb{F} is an extension of \mathbb{Q}_2 since it is local and the characteristic 2 for \mathbb{F} itself is excluded at the outset) then $\Gamma_1(4, \mathcal{O})$ is a proper subgroup of $SL_2(\mathcal{O})$, and in general Theorem 6.3 cannot be extended to the full group $SL_2(\mathcal{O})$. This observation can be demonstrated in \mathbb{Q}_2 itself: $-I$ has order 2 in $SL_2(\mathbb{Z}_2)$, but $(-1, -1)_{\mathbb{Q}_2} = -1$ implies that both elements lying over $-I$ have order 4 in $Mp_2(\mathbb{Q}_2)$. Hence no lift of $SL_2(\mathbb{Z}_2)$ into $Mp_2(\mathbb{Q}_2)$ exists. This also shows that even the inverse image of $SL_2(\mathbb{Z})$ in $Mp_2(\mathbb{Q}_2)$ is a non-split double cover of $SL_2(\mathbb{Z})$ (see also Theorem 6.6 below). One can show that the same argument holds for every odd-dimensional extension of \mathbb{Q}_2 : The square of Eq. (28) and the fact that $\gamma(x^2)^4 = (-1, -1)_{\mathbb{F}}$ (by Eq. (2)) show that odd-dimensional quadratic forms have Weil indices of order dividing 4 or exactly 8 according to whether $(-1, -1)_{\mathbb{F}}$ is 1 or -1 , and the parity of the dimension of such a vector space over \mathbb{Q}_2 is the same as that of the degree of the extension \mathbb{F}/\mathbb{Q}_2 . It is likely that this occurs for every extension of \mathbb{Q}_2 (see the remark at the end of Section 9).

What we can show, however, is the following

Proposition 6.5. *The image of the lift ι of $\Gamma(4, \mathcal{O})$ is a normal subgroup of the double cover $Mp_2(\mathcal{O})$ of $SL_2(\mathcal{O})$.*

Proof. When $2 \in \mathcal{O}^*$ we have $\Gamma(4, \mathcal{O}) = SL_2(\mathcal{O})$ and all that separates the image of ι from $Mp_2(\mathcal{O})$ is the central element $(1, -1)$. Hence the claim is obvious in this case. Next we assume that the residue characteristic of \mathbb{F} is 2, so that for any matrix in $\Gamma(4, \mathcal{O})$ (with the usual entries) both a and d are invertible. Since $(1, -1)$ is central, we can consider the conjugation as if it is by elements of $SL_2(\mathcal{O})$. We extend the notation T^x from $x \in \mathbb{Z}$ (i.e., powers of T) to the corresponding matrices for any $x \in \mathcal{O}$, and as in Lemma 3.1 we see that a lower triangular unipotent matrix with lower left entry c is $ST^{-c}S^{-1}$. It follows from the SL_2 condition that $SL_2(\mathcal{O})$ is generated by S and the elements T^b with $b \in \mathcal{O}$: If $c \in \mathcal{O}^*$ then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a-1}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d-1}{c} \\ 0 & 1 \end{pmatrix},$$

while if c is not invertible then a and d are invertible and both SA and AS have invertible lower left entry. Hence it is enough to verify that conjugation by T^x and S preserve the image of the lift of $\Gamma(4, \mathcal{O})$. We continue to use the notation $\iota(A) = (A, \varepsilon_A)$ from the proof of Theorem 6.3. All the "cases" appearing below correspond to Eq. (4).

Now, for any $A \in SL_2(\mathbb{F})$ and $x \in \mathcal{O}$ we have $\sigma(T^x, A) = \sigma(A, T^x) = 1$. This follows from the first case if $c = 0$ and from the second and third cases

respectively if $c \neq 0$, since the diagonal entries of T^x are 1 (compare to Lemma 4.3 of [Str]). The conjugation $T^x A T^{-x}$ leaves the c entry unchanged and adds to the a and d entries multiples of c (hence of $\pi^{v(c)}$). Using part (iii) of Lemma 6.1 for the $c \neq 0$ case (which applies since $A \in \Gamma(4, \mathcal{O})$) we find that $\varepsilon_{T^x A T^{-x}} = \varepsilon_A$. Hence conjugation by T^x preserves the image $\iota(\Gamma(4, \mathcal{O}))$. For conjugation by S , we first note that $\sigma(S, S^{-1}) = \sigma(S^{-1}, S) = 1$ (by the fourth case), so that in “conjugation by $S \in SL_2(\mathcal{O})$ ” we must consider S and S^{-1} as coming with the same metaplectic sign. Therefore we have to check that ε_A and the signs coming from the cocycle σ of Eq. (4) combine to give $\varepsilon_{SAS^{-1}}$. The results are as follows (compare Lemma 4.4 of [Str]): $\sigma(S, A)$ is 1 if $c = 0$ and $(-c, a)_{\mathbb{F}}$ otherwise (third and fifth cases respectively), while $\sigma(SA, S^{-1})$ is 1 if $b = 0$ and $(a, b)_{\mathbb{F}}$ otherwise (fourth and fifth cases respectively). Alternatively, $\sigma(A, S^{-1})$ is $(d, -1)_{\mathbb{F}}$ if $c = 0$ and $(c, d)_{\mathbb{F}}$ otherwise (second and fifth cases respectively), while $\sigma(S, AS^{-1})$ is $(-1, d)_{\mathbb{F}}$ if $b = 0$ and $(d, -b)_{\mathbb{F}}$ (fourth and fifth cases respectively). Summarizing, the bilinearity and symmetry of the Hilbert symbol show that the product is 1 if $c = b = 0$, $(a, -c)_{\mathbb{F}} = (d, -c)_{\mathbb{F}}$ if $b = 0 \neq c$, $(a, b)_{\mathbb{F}} = (d, b)_{\mathbb{F}}$ if $b \neq 0 = c$, and $(a, -bc)_{\mathbb{F}} = (d, -bc)_{\mathbb{F}}$ if $bc \neq 0$ (all the equalities follow from the SL_2 condition and part (iii) of Lemma 6.1). By definition, ε_A is 1 for $c = 0$ and $(a, \pi^{v(c)})_{\mathbb{F}} = (d, \pi^{v(c)})_{\mathbb{F}}$ otherwise, while $\varepsilon_{SAS^{-1}}$ is 1 for $b = 0$ and $(a, \pi^{v(b)})_{\mathbb{F}} = (d, \pi^{v(b)})_{\mathbb{F}}$. The fact that a and d are congruent to 1 modulo 4 (hence invertible) implies, by part (ii) of Lemma 6.1, that the cocycle gives exactly the required difference between the signs. We remark that the only use of 2 being not invertible is when claiming that a and d are in \mathcal{O}^* , so that the same proof extends to the other cases as well, with two additional cases that have to be checked. However, as indicated in the previous paragraph, the centrality of $(1, -1)$ suffices to ensure that the result holds in all cases.

The proof of the proposition is now complete. \square

For $\mathbb{F} = \mathbb{Q}_2$ we can obtain the lift of $\Gamma(4, \mathbb{Z}_2)$ as the kernel of the representation of $Mp_2(\mathbb{Z}_2)$ on a discriminant form of cardinality 2 which comes from a lattice of rank 1 when using an isomorphism similar to the one appearing in Lemma 2.2, as well as the tools of Section 8. This observation suggests another proof of Proposition 6.5 for this case, which can probably also be extended to the general case (although some technical difficulties arise—see Section 9). We discuss this point a bit further in Section 8.

6.4 Comparing the Real and 2-adic $Mp_2(\mathbb{Z})$

We now consider the two “metaplectic groups over \mathbb{Z} ”: the one embedded in $Mp_2(\mathbb{R})$ as in Section 1, and the other embedded in $Mp_2(\mathbb{Q}_2)$ (whose closure there is the non-trivial double cover $Mp_2(\mathbb{Z}_2)$ of $SL_2(\mathbb{Z}_2)$). Our main object of interest, the representation ρ_M of Section 1, is defined using the former group, while the representation ρ_{M_2} is defined on (the closure of) the latter. We have identified (A, ε) in the “abstract” $Mp_2(\mathbb{R})$ with $(A, \varepsilon \sqrt{j(A, \tau)})$ in the “modular” $Mp_2(\mathbb{R})$ where the argument of $\sqrt{j(A, \tau)}$ is assumed to be in $[-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore

we use the “abstract” notation only for elements in $Mp_2(\mathbb{Q}_2)$. The relation between these two double covers of $SL_2(\mathbb{Z})$ is given in the following

Theorem 6.6. *The map i from the “abstract” $Mp_2(\mathbb{Z}) \subseteq Mp_2(\mathbb{R})$ to $Mp_2(\mathbb{Q}_2)$ which takes an element $(A, \varepsilon \sqrt{j(A, \tau)})$ in the former to (A, ε) if $c = 0$ and to $(A, (\frac{a}{c_2})\varepsilon) = (A, (\frac{d}{c_2})\varepsilon)$ if $c \neq 0$ is a group injection.*

Proof. This map is clearly injective, and the fact that $(\frac{a}{c_2}) = (\frac{d}{c_2})$ is evident from $ad \equiv 1 \pmod{c}$ (this holds as well if a or d vanish since then $c = \pm 1$ and the Legendre symbol over it is 1). Similarly to the proof of Theorem 6.3 we write $i(A, \varepsilon \sqrt{j(A, \tau)}) = (A, \varepsilon \delta_A)$ (since the additional sign depends only on A and not on ε). In order to verify that this is a group homomorphism we need to follow the proofs of Theorems 5.2 and 6.3 and check the sign of the product according to the 5 cases of Eq. (4). Moreover, since in both groups the signs just multiply (up to an additional sign coming from the matrices), it suffices to verify this only for the products of $(A, \sqrt{j(A, \tau)})$ and $(B, \sqrt{j(B, \tau)})$. Now, the product in $Mp_2(\mathbb{R})$ is based on Eq. (4) with $\mathbb{F} = \mathbb{R}$, where $(x, y)_{\mathbb{R}} = (-1)^{\sigma(x)\sigma(y)}$, while in $Mp_2(\mathbb{Q}_2)$ it is based on Eq. (4) with $\mathbb{F} = \mathbb{Q}_2$, and

$$(x, y)_{\mathbb{Q}_2} = (-1)^{\varepsilon(x_2)\varepsilon(y_2)} \left(\frac{2}{x_2}\right)^{v_2(y)} \left(\frac{2}{y_2}\right)^{v_2(x)}.$$

Thus one has to verify that in each of the 5 cases of Eq. (4), the difference between the Hilbert symbols is exactly $\delta_A \delta_B \delta_{AB}$. In order to do so we note that for any $x \in \mathbb{Q}^*$ we have $\sigma(x) = \sigma(x_2)$, implying that the formulae for the Hilbert symbols over \mathbb{R} and \mathbb{Q}_2 give rise to the useful equality stating that for any coprime nonzero integers r and s we have

$$\left(\frac{r}{s_2}\right) \left(\frac{s}{r_2}\right) = (r, s)_{\mathbb{Q}_2} (r, s)_{\mathbb{R}}. \quad (5)$$

Indeed, removing the powers of 2 from both sides and using the quadratic reciprocity law (extended to include negative odd numbers as well) verifies the result. In fact, Eq. (5) is a form of the reciprocity law for Hilbert symbols, under the assumption that r and s are coprime integers.

We return to consider each case of Eq. (4) separately. In the first case we have $a = d = \pm 1$, $e = h = \pm 1$, and $\delta_A = \delta_B = \delta_{AB} = 1$. Since Eq. (5) inverts the product of the Hilbert symbols in question to a product of Legendre symbols over ± 1 , the assertion holds for this case. In the second case $\delta_A = 1$ and $d = \pm 1$, so that the Legendre symbols over g_2 and over dg_2 are the same. This implies $\delta_B \delta_{AB} = (\frac{d}{g_2})$ (also when $h = 0$ —Legendre symbols over ± 1 are 1), and since $(\frac{g}{d_2}) = 1$ this case is also complete by Eq. (5). The third case is very similar: We have $\delta_B = 1$, $\delta_A \delta_{AB} = (\frac{e}{c_2})$ (also when $a = 0$), and the result follows from Eq. (5) since $(\frac{c}{e_2}) = 1$. In the fourth case the $SL_2(\mathcal{O})$ conditions imply that $c = \pm g$, so that the Legendre symbol over g_2 is the same as that over c_2 . We also have $\delta_{AB} = 1$, and the equality $e = -\frac{dg}{c}$ implies $\delta_A \delta_B = (\frac{-g/c}{c_2}) = (\frac{\pm 1}{c_2})$

(also if $d = e = 0$, for the usual reason). On the other hand, for any Hilbert symbol we have $(c, -c) = 1$ (and bilinearity), so that the product of the Hilbert symbols in Eq. (4) can be replaced by the product of the Hilbert symbols of $-c$ and ∓ 1 . According to Eq. (5) this gives $\left(\frac{\mp 1}{-c_2}\right)\left(\frac{-c}{\mp 1}\right)$, and since $\left(\frac{c}{\mp 1}\right) = 1$ and $\left(\frac{\mp 1}{-c_2}\right) = \left(\frac{\mp 1}{c_2}\right)$ in our convention, the assertion is proved for this case as well.

In the fifth case we denote $ce + dg$ by x and use Lemma 6.2. If $d = 0$ then $c = \pm 1$, $x = \pm e$, $\delta_A = 1$, and Eq. (3a) shows that $\delta_{AB} = \left(\frac{\mp g}{e_2}\right)$. Eq. (5) then shows that the product $\delta_B \delta_{AB}$ is $(e, \mp g)_{\mathbb{R}}(e, \mp g)_{\mathbb{Q}_2}$, as desired (since $\mp g = -cg$). Similarly, for $e = 0$ we have $g = \pm 1$, $x = \pm d$, $\delta_B = 1$, $\delta_{AB} = \left(\frac{\mp c}{d_2}\right)$ by Eq. (3a), and the product $\delta_A \delta_{AB}$ is the desired $(d, \mp c)_{\mathbb{R}}(d, \mp c)_{\mathbb{Q}_2}$ by Eq. (5). Finally, if d and e are both non-zero then $\gcd\{c, d\} = \gcd\{e, g\} = 1$ implies that $\gcd\{d, x\} = \gcd\{e, x\} = \gcd\{d, e\}$. We call this common value y , and it is coprime to c and g (since it divides d and e). Hence we write $d = y \frac{d}{y}$ in δ_A and $e = y \frac{e}{y}$ in δ_B , and we separate the expression for δ_{AB} as the product of two Legendre symbols, one over y_2 and the other over $\frac{x_2}{y_2}$. Using Eq. (3a) for the former and Eq. (3b) divided by y for the latter allows us to write δ_{AB} as $\left(\frac{-cg}{y_2}\right)\left(\frac{de/y^2}{x_2/y_2}\right)$. This yields the equality

$$\delta_A \delta_B \delta_{AB} = \left(\frac{d/y}{c_2 x_2 / y_2}\right) \left(\frac{e/y}{g_2 x_2 / y_2}\right) \left(\frac{-cg}{y_2}\right) \left(\frac{y}{c_2 g_2}\right).$$

The product of the last two Legendre symbols is $(y, -cg)_{\mathbb{R}}(y, -cg)_{\mathbb{Q}_2}$ by Eq. (5). On the other hand, applying Eq. (5) to each of the other Legendre symbols yields the product

$$\left(\frac{d}{y}, \frac{cx}{y}\right)_{\mathbb{R}} \left(\frac{d}{y}, \frac{cx}{y}\right)_{\mathbb{Q}_2} \left(\frac{e}{y}, \frac{gx}{y}\right)_{\mathbb{R}} \left(\frac{e}{y}, \frac{gx}{y}\right)_{\mathbb{Q}_2} \left(\frac{cx/y}{d_2/y_2}\right) \left(\frac{gx/y}{e_2/y_2}\right).$$

The bilinearity and symmetry of the Hilbert symbols imply that the product of the Hilbert symbols appearing in the latter expression equals

$$(d, cx)_{\mathbb{R}}(d, cx)_{\mathbb{Q}_2}(e, gx)_{\mathbb{R}}(e, gx)_{\mathbb{Q}_2}(y, cdx)_{\mathbb{R}}(y, cdx)_{\mathbb{Q}_2}(y, egxy)_{\mathbb{R}}(y, egxy)_{\mathbb{Q}_2}.$$

The first four Hilbert symbols appear in the desired expression appearing in Lemma 6.2 over \mathbb{R} and \mathbb{Q}_2 , while the Hilbert symbols involving y combine with $(y, -cg)_{\mathbb{R}}(y, -cg)_{\mathbb{Q}_2}$ from above to yield $(y, -de)_{\mathbb{R}}(y, -de)_{\mathbb{Q}_2}$ (as all the squares can be omitted). It therefore remains to prove that multiplying the latter Hilbert symbols by $\left(\frac{cx/y}{d_2/y_2}\right)\left(\frac{gx/y}{e_2/y_2}\right)$ yields the remaining expression $(d, e)_{\mathbb{R}}(d, e)_{\mathbb{Q}_2}$ from Lemma 6.2 over \mathbb{R} and \mathbb{Q}_2 . But the observations

$$\frac{cx}{y} \equiv \frac{c^2 e}{y} \left(\text{mod } \frac{d}{y}\right), \quad \frac{gx}{y} \equiv \frac{dg^2}{y} \left(\text{mod } \frac{e}{y}\right), \quad \text{and} \quad \left(\frac{c^2}{d_2/y_2}\right) = \left(\frac{g^2}{e_2/y_2}\right) = 1$$

imply that the Legendre symbols in question reduce to $\left(\frac{e/y}{d_2/y_2}\right)\left(\frac{d/y}{e_2/y_2}\right)$, whose product is $\left(\frac{d}{y}, \frac{e}{y}\right)_{\mathbb{R}}\left(\frac{d}{y}, \frac{e}{y}\right)_{\mathbb{Q}_2}$ by Eq. (5). Multiplying by the remaining Hilbert symbols $(y, -de)_{\mathbb{R}}(y, -de)_{\mathbb{Q}_2}$ and using the bilinearity and symmetry of the

Hilbert symbols yields the desired expression multiplied by $(y, -y)_{\mathbb{R}}(y, -y)_{\mathbb{Q}_2}$. As the latter Hilbert symbols are known to equal 1, the assertion follows.

This completes the proof of the theorem. \square

In particular, the inverse image of $SL_2(\mathbb{Z})$ in $Mp_2(\mathbb{Q}_2)$ is the image of i , hence is isomorphic to $Mp_2(\mathbb{Z})$. The fact that the inverse image of $SL_2(\mathbb{Z})$ in $Mp_2(\mathbb{Q}_2)$ is a non-trivial double cover of $SL_2(\mathbb{Z})$ has not been used in the proof of Theorem 6.6, hence we could have obtained the non-triviality of the latter double cover of $SL_2(\mathbb{Z})$ as an immediate corollary of Theorem 6.6. In any case, it follows from Theorem 6.6 that the double cover of $SL_2(\mathbb{Z})$ inside $Mp_2(\mathbb{R})$ and the one inside $Mp_2(\mathbb{Q}_2)$ are the same (non-trivial) double cover.

The following result turns out useful:

Corollary 6.7. *Let V be a vector space of dimension m over \mathbb{Q}_2 . Keeping the same choice of the square root of $j(A, \tau)$ for $A \in SL_2(\mathbb{Z})$, the map which takes $(A, \varepsilon \sqrt{j(A, \tau)})$ to $\varepsilon^m (\frac{a}{c_2})^m \gamma(cf) \mathbf{r}_0(A) = \varepsilon^m (\frac{d}{c_2})^m \gamma(cf) \mathbf{r}_0(A)$ if $c \neq 0$ and to $\varepsilon^m \gamma(af) \gamma(f) \mathbf{r}_0(A)$ if $c = 0$ is a representation of $Mp_2(\mathbb{Z})$ on $L^2(V)$, which is faithful for odd m and factors through a faithful representation of $SL_2(\mathbb{Z})$ for even m , and preserves the dense subspace $\mathcal{S}(V)$.*

Proof. Compose i of Theorem 6.6 with ρ_{V/\mathbb{Q}_2} of Theorem 5.2. \square

In particular, checking the actions of T and S we find that the representation of $Mp_2(\mathbb{Z})$ obtained from the \mathbb{Q}_2 -lattice V by the process from Section 1 coincides with the representation described in Corollary 6.7.

We remark that one can prove Theorem 6.6 and Corollary 6.7 from Theorem 6.3 and Corollary 6.4 using Adélic considerations, the Weil Reciprocity Law, and the fact that the product of the Hilbert symbols over all the places of \mathbb{Q} always equals 1. The discussion of the global theory in [W] and [Ge] provides the ideas on which such a proof can be based. Moreover, such a proof will be shorter than the one given here. However, our elementary proof here is more in line with the approach of the rest of this paper.

Proposition 6.5 and Theorem 6.6 combine to give the following

Corollary 6.8. *The map from $\Gamma_1(4)$ into the “modular” $Mp_2(\mathbb{Z})$ defined by taking $A \in \Gamma_1(4)$ to $(A, (\frac{2}{a})^{v_2(c)} (\frac{a}{c_2})) = (A, (\frac{2}{a})^{v_2(c)} (\frac{d}{c_2}))$ is a lift, and the image of the restriction of this lift to $\Gamma(4)$ is normal in $Mp_2(\mathbb{Z})$.*

Proof. Take $\mathbb{F} = \mathbb{Q}_2$, hence $\mathcal{O} = \mathbb{Z}_2$, in Theorem 6.3 and Proposition 6.5, and restrict to matrices in $SL_2(\mathbb{Z})$. The proof of Theorem 6.3 shows that by part (ii) of Lemma 6.1, we can write ε_A appearing in $\iota(A)$ for A with $c \neq 0$ as $(a, c)_{\mathbb{Q}_2} = (d, c)_{\mathbb{Q}_2}$. According to Proposition 6.5 the image of $\Gamma(4)$ by this map is normal in $Mp_2(\mathbb{Z})$ which lies inside $Mp_2(\mathbb{Q}_2)$ (or $Mp_2(\mathbb{Z}_2)$). Moreover, as $A \in \Gamma_1(4)$ these Hilbert symbols reduce to $(\frac{2}{a})^{v_2(c)} = (\frac{2}{d})^{v_2(c)}$. Composing with i^{-1} , the inverse of the map i from Theorem 6.6 (which is an isomorphism onto its image) gives a lift of $\Gamma_1(4)$ into the “modular” $Mp_2(\mathbb{Z})$, which is given explicitly by the asserted formula (the result holds also when $c = 0$, since then

$a = d = 1$ and all the Legendre symbols are 1). This image of $\Gamma(4)$ has been shown to be normal. This proves the corollary. \square

Eq. (5), together with the fact that a and d are odd and the explicit expression for the real Hilbert symbol, allow us to write the sign in this lift $i^{-1}\iota$ as $(\frac{c}{a})(-1)^{\sigma(a)\sigma(c)}$, or equivalently as $(\frac{c}{d})(-1)^{\sigma(d)\sigma(c)}$. The latter coincides with the one denoted s in [BS], and with the one appearing in Lemma 5.3 of [B3] for $\Gamma_1(4)$ (note the convention difference for Legendre symbols).

The observation that for any vector space over \mathbb{F} we obtain either a representation of $SL_2(\mathbb{F})$ or of the *same* double cover $Mp_2(\mathbb{F})$ is not coincidental. Indeed, for $\mathbb{F} = \mathbb{R}$ we know that the fundamental group of the Lie group $SL_2(\mathbb{R})$ is infinite cyclic, hence has only one subgroup of index 2. Therefore this group has only one non-trivial double cover. For the other fields we have Proposition 2.3 of [Ge] (whose proof can probably be extended without changes from the characteristic 0 case to all characteristics other than two). This Proposition states that the second cohomology group $H^2(SL_2(\mathbb{F}), \{\pm 1\})$ is $\{\pm 1\}$, so that $SL_2(\mathbb{F})$ has only one non-trivial double cover up to isomorphism. Also, $SL_2(\mathbb{C})$ has no non-trivial double covers (or, more generally, no non-trivial covers at all), since it is a simply connected Lie group. On the other hand, for $SL_2(\mathcal{O})$ in the odd residue characteristic we found only trivial double covers, while for each of $SL_2(\mathbb{Z})$ and $SL_2(\mathbb{Z}_2)$ we found one we found only one non-trivial double cover. It would be interesting to obtain results about the H^2 groups in this case, i.e., of SL_2 of the rings of integers inside local and global fields with coefficients in $\{\pm 1\}$.

7 Evaluation for Local Operators

In this Section we evaluate, for a non-archimedean \mathbb{F} , the operators $\mathbf{r}_0(A)$ for a matrix $A \in SL_2(\mathbb{F})$ with integral entries, on certain Schwartz functions under some assumptions on the character λ chosen on $(\mathbb{F}, +)$. The formulae presented here form the technical heart of the derivation of our main result in the following Section.

7.1 First Formulae

Let \mathbb{F} be a non-archimedean local field of characteristic $\neq 2$, let \mathcal{O} be its ring of integers, let π be a uniformizer, let v be the normalized valuation, and denote the cardinality of the residue field by q . Take a character λ on $(\mathbb{F}, +)$ which satisfies the property that $\lambda(x\mathcal{O}) = 1$ if and only if $x \in \mathcal{O}$. We adopt the following notation for characteristic functions: If Y is a subset of some set X then the *characteristic function* of Y in X , i.e., the function on X which attains 1 on elements of Y and 0 on the other elements of X , will be denoted E_Y . We shall use this notation only in situations where the ambient set X is clear from the context, so that no confusion arises.

Let M be an even \mathcal{O} -lattice of rank m . We consider the characteristic function $E_{M+\gamma}$ of a coset $M + \gamma$ for some $\gamma \in D_M$ inside the rank m \mathbb{F} -lattice $M_{\mathbb{F}}$.

This function is in $\mathcal{S}(M_{\mathbb{F}})$. We now evaluate the action of $\mathbf{r}_0(A)$, for $A \in SL_2(\mathcal{O})$ with the usual entries, on $E_{M+\gamma}$, i.e., $\mathbf{r}_0(A)E_{M+\gamma}$. We further note that in this case M/cM is a finite group, being of finite length over a ring with a unique simple module which is finite. Hence the sums discussed below are finite and no convergence issues arise. The first step of evaluating $\mathbf{r}_0(A)E_{M+\gamma}$ is described in the following

Proposition 7.1. *If $c = 0$ then $\mathbf{r}_0(A)E_{M+\gamma}$ gives $\lambda(bd\frac{\gamma^2}{2})E_{M+d\gamma}$. If $c \neq 0$ then it gives*

$$\frac{1}{q^{mv(c)/2}\sqrt{\Delta_M}} \sum_{\delta \in D_M} \left[\sum_{\eta \in M/cM} \lambda\left(\frac{a}{c} \frac{(\delta + \eta - d\gamma)^2}{2}\right) \right] \lambda\left(b(\gamma, \delta) - bd\frac{\gamma^2}{2}\right) E_{M+\delta}.$$

Proof. If $c = 0$ then the formula for $\mathbf{r}_0(A)E_{M+\gamma}(x)$ with $x \in M_{\mathbb{F}}$ gives the coefficient $\lambda(ab\frac{x^2}{2})$ times $E_{M+\gamma}(ax)$, since $\det a = a^m$ has valuation 0. Since $ad = 1$ and $d \in \mathcal{O}^*$, the latter expression reduces to $E_{M+d\gamma}$, hence is non-zero only for $x = d\gamma + u$ with $u \in M$. But then $\frac{x^2}{2}$ is congruent to $d^2\frac{\gamma^2}{2}$ modulo \mathcal{O} (since M is even), and $ad = 1$ implies $abd^2 = bd$. This covers the case $c = 0$ since $\lambda(\mathcal{O}) = 1$.

If $c \neq 0$ then the formula for $\mathbf{r}_0(A)E_{M+\gamma}(x)$ for some $x \in M_{\mathbb{F}}$ contains the coefficient $\frac{1}{\sqrt{|\det c|_{\mathbb{F}}}}$, which becomes $q^{mv(c)/2}$, and we have to evaluate the associated integral. Any element in $M+\gamma$ can be written as $\gamma + \eta + v$ with unique $\eta \in M/cM$ and $v \in cM$, which implies that $E_{M+\gamma} = \sum_{\eta \in M/cM} E_{cM+\gamma+\eta}$. For each η the substitution $y = \gamma + \eta + v$ now yields

$$\begin{aligned} & \int_{M+\gamma} \lambda\left(\frac{d}{c} \frac{y^2}{2} - \frac{(y, x)}{c} + \frac{a}{c} \frac{x^2}{2}\right) = \\ &= \sum_{\eta \in M/cM} \int_{cM} \lambda\left(\frac{d}{c} \frac{(\gamma + \eta)^2}{2} + \frac{d}{c}(\gamma + \eta, v) + \frac{d}{c} \frac{v^2}{2} - \frac{(\gamma + \eta, x)}{c} - \frac{(v, x)}{c} + \frac{a}{c} \frac{x^2}{2}\right). \end{aligned}$$

Substituting $v = cu$ with $u \in M$ shows that for any η the integral becomes

$$q^{-mv(c)} \int_M \lambda\left(\frac{d}{c} \frac{(\gamma + \eta)^2}{2} + \overbrace{d(\gamma + \eta, u)} + \overbrace{cd\frac{u^2}{2}} - \frac{(\gamma + \eta, x)}{c} - (u, x) + \frac{a}{c} \frac{x^2}{2}\right).$$

Both over-braced elements are in \mathcal{O} , so that the integral reduces to a constant times $\int_M \lambda(- (x, u)) du$. The property of λ implies that the latter integral yields the measure of M (which is $\frac{1}{\sqrt{\Delta_M}}$ by the normalization of the measure) if $x \in M^*$, and 0 otherwise. Hence $\mathbf{r}_0(A)E_{M+\gamma}(x)$ does not vanish only for $x \in M^*$. Consider such x , and then $x \in M + \delta$ for some $\delta \in D_M$. By writing $x = \delta + w$ with $w \in M$ we obtain the total expression for $\mathbf{r}_0(A)E_{M+\gamma}(\delta + w)$ as a product of a well-understood constant and a Gauss sum, namely

$$\frac{1}{q^{mv(c)/2}\sqrt{\Delta_M}} \sum_{\eta \in M/cM} \lambda\left(\frac{d}{c} \frac{(\gamma + \eta)^2}{2} - \frac{(\gamma + \eta, \delta + w)}{c} + \frac{a}{c} \frac{(\delta + w)^2}{2}\right).$$

Now, multiplication by a is injective on M/cM since a and c are coprime. We can thus replace η by $a\eta$, and in fact we replace η by $aw - a\eta$ (which is possible since $w \in M$). We then expand all terms in parenthesis, and write $\frac{ad}{c} = \frac{1}{c} + b$ and $\frac{a^2d}{c} = \frac{a}{c} + ab$. Now, $b(\gamma, w)$, $ab\frac{w^2}{2}$, $ab(\eta, w)$, $b(\gamma, \eta)$, and $ab\frac{\eta^2}{2}$ are all in \mathcal{O} , hence collecting the remaining expressions yields $\mathbf{r}_0(A)E_{M+\gamma}(\delta+w)$ in the form

$$\frac{1}{q^{mv(c)/2}\sqrt{\Delta_M}} \sum_{\eta \in M/cM} \lambda \left(\frac{d}{c} \frac{\gamma^2}{2} - \frac{(\gamma, \delta + \eta)}{c} + \frac{a}{c} \frac{(\delta + \eta)^2}{2} \right). \quad (6)$$

An important advantage of Eq. (6) is that it is independent of w , which means that $\mathbf{r}_0(A)E_{M+\gamma}$ is a linear combination of the functions $E_{M+\delta}$ with $\delta \in D_M$. We now expand $\frac{1}{c} = \frac{ad}{c} - b$ and $\frac{d}{c} = \frac{ad^2}{c} - bd$, and gathering the elements containing a turns the Gauss sum in Eq. (6)

$$\sum_{\eta \in M/cM} \lambda \left(\frac{a}{c} \frac{(\delta + \eta - d\gamma)^2}{2} + b(\gamma, \delta + \eta) - bd\frac{\gamma^2}{2} \right).$$

Recalling that $b(\gamma, \eta) \in \mathcal{O}$, the proof of the proposition is now complete. \square

We note that the Gauss sum in Proposition 7.1 is well-defined: Changing η by an element of cM gives the same summand, and changing γ or δ by an element of M is equivalent to changing the summation index η . We also note that when $c \in \mathcal{O}^*$ Eq. (6) agrees with our form of Eq. (16) of [W] for the group D_M .

We recall the identification of the space $\mathbb{C}[M^*/M]$ with the subspace of $\mathcal{S}(M_{\mathbb{F}})$ spanned by the functions $E_{M+\gamma}$ with $\gamma \in D_M$ as described in Section 2 (in that Section we considered only $\mathbb{F} = \mathbb{Q}_p$, but the generalization is immediate). Thus, Proposition 7.1 provides an alternative proof of Lemma 2.2: ρ_{M_p} is \mathbf{r}_0 up to multiplication by a constant, and the character χ_p on \mathbb{Q}_p has kernel \mathbb{Z}_p , so that it can be taken as λ here. It turns out convenient to write the results in terms of the space $\mathbb{C}[M^*/M]$, and we denote this space again by V_{ρ_M} (as it is the space on which a representation suitably denoted ρ_M acts). Using the previous notation for the usual canonical basis, we find

Corollary 7.2. *If $c = 0$ then $\mathbf{r}_0(A)e_{\gamma}$ equals $\lambda(bd\frac{\gamma^2}{2})e_{d\gamma}$, while for $c \neq 0$ it equals*

$$\frac{1}{q^{mv(c)/2}\sqrt{\Delta_M}} \sum_{\beta \in D_M} \left[\sum_{\eta \in M/cM} \lambda \left(\frac{a}{c} \frac{\eta^2}{2} + a \frac{(\beta, \eta)}{c} \right) \right] \lambda \left(\frac{a}{c} \frac{\beta^2}{2} + b(\gamma, \beta) + bd\frac{\gamma^2}{2} \right) e_{\beta+d\gamma}.$$

Proof. Translating Proposition 7.1 into the V_{ρ_M} terminology yields the first assertion. To prove the second assertion we write $\delta = \beta + d\gamma$ and note that $b(\gamma, d\gamma) - bd\frac{\gamma^2}{2}$ gives $+bd\frac{\gamma^2}{2}$. Expanding $\frac{(\beta+\eta)^2}{2}$ in the sum over η completes the proof of the corollary. \square

We note that the form of $\mathbf{r}_0(A)e_{\gamma}$ with $c \neq 0$ that we derived in Corollary 7.2 is more convenient than the direct translation of Proposition 7.1 since the Gauss sum is now independent of γ .

7.2 Subsets of Discriminant Forms

Let M be an even lattice over an integral domain R whose fraction field \mathbb{K} is not of characteristic 2, and let c be an element of R . Multiplication by c provides a natural map $c : D_M \rightarrow D_M$. Following [Sche] and [Str] we denote its kernel by $D_{M,c}$ and its image by D_M^c , and it is not hard to see that in the non-degenerate \mathbb{K}/R -valued pairing on D_M , these two subgroups are orthogonal complements. Indeed, they are mutually orthogonal and lie in the natural exact sequence. Moreover, the map which takes $\mu \in D_{M,c}$ to $c\frac{\mu^2}{2} + (\beta, \mu) \in \mathbb{K}/R$ is linear on $D_{M,c}$ for any $\beta \in D_M$, and we denote by D_M^{c*} the set of those $\beta \in D_M$ for which this map is identically 0 in \mathbb{K}/R . This set is a coset of the subgroup D_M^c inside D_M —see Proposition 2.1 of [Sche] for the case $R = \mathbb{Z}$, and the proof holds equally well for the more general setting. We choose an element x_c in the coset (in future applications we shall specify the choice), so that any $\beta \in D_M^{c*}$ is $x_c + c\alpha$ for some $\alpha \in D_M$ which is well-defined up to $D_{M,c}$. The element $\frac{\beta^2}{2} = c\frac{\alpha^2}{2} + (x_c, \alpha)$ is well-defined in \mathbb{K}/R , i.e., independent of the choice of α , since $x_c \in D_M^{c*}$ (this is Proposition 2.2 of [Sche], which generalizes as well). We note however that this element depends on the choice of x_c , but we consider x_c as a pre-fixed element of D_M . It is easy to see that if $2 \in \mathcal{O}^*$ then $D_M^{c*} = D_M^c$ (since by definition $c\mu^2 \in \mathcal{O}$), so that the natural choice in this case is to take $x_c = 0$. It is evident from the definition that $D_{M \oplus N} = D_M \oplus D_N$ (with both direct sum being orthogonal), and then an element in $D_{M \oplus N}$ lies in $D_{M \oplus N, c}$, $D_{M \oplus N}^c$, or $D_{M \oplus N}^{c*}$ if and only if its components in this decomposition lie in the corresponding sets for M and N . We remark that all these observations hold also for $c = 0$, where $D_{M,0} = D_M$, $D_M^0 = D_M^{0*} = \{0\}$, $x_0 = 0$, and $\frac{\beta_0}{2} = 0 \in \mathbb{K}/R$.

In the applications we are interested in $R = \mathcal{O}$, the ring of integers in a local or global field of characteristic $\neq 2$. In this case D_M (hence also its subsets) is finite, and we define $\Delta_{M,c}$ to be the cardinality of $D_{M,c}$ (as in Theorem 4.3 above). We note that if R is the ring of integers in a global field, then we can decompose D_M as $\bigoplus_P D_{M_P}$ for all primes P in R , and it is clear from the definition that for any $c \in R$, an element $\beta \in D_M$ lies in $D_{M,c}$, D_M^c , or D_M^{c*} if and only if its image in D_{M_P} (i.e., its P -part) lies in the corresponding set for any prime P (this structure is similar to the orthogonal direct sum decomposition claim of the previous paragraph).

Returning to the case where \mathbb{F} is local and non-archimedean with ring of integers \mathcal{O} , we now obtain

Lemma 7.3. *The Gauss sum in Corollary 7.2 vanishes for $\beta \notin D_M^{c*}$.*

Proof. The statement of the lemma is trivial if $c \in \mathcal{O}^*$, since then $D_M^{c*} = D_M$ (and the Gauss sum is just 1). Assume now that c is not invertible, whence a is invertible since a and c are coprime. Now, take $\rho \in M^*$ such that $c\rho \in M$ (i.e., the image of ρ in D_M lies in $D_{M,c}$), and change the summation index η to $\eta + c\rho$. Then since $a(\eta, \rho) \in \mathcal{O}$ we find that this operation multiplies the Gauss sum by $\lambda(ac\frac{\rho^2}{2} + a(\beta, \rho))$. For $\beta \in D_M^{c*}$ this term is always 1, so that no difficulty arises. Otherwise, the fact that $\lambda(x\mathcal{O}) = 1$ only for $x \in \mathcal{O}$ and $a \in \mathcal{O}^*$

allows us to obtain an element ρ such that the multiplier is not 1, which implies that the sum vanishes as required. This proves the lemma. \square

This result is closely related to Proposition 3.8 of [Sche] and to Lemma 3.1 of [B3], even though different Gauss sums are considered here.

We now choose an element x_c in M^* whose image in D_M lies in the coset D_M^{c*} , and obtain the following

Corollary 7.4. *If $c \neq 0$ then $\mathbf{r}_0(A)e_\gamma$ is*

$$\frac{\lambda\left(\frac{a}{c}\frac{x_c^2}{2}\right)}{q^{mv(c)/2}\sqrt{\Delta_M}} \sum_{\eta \in M/cM} \lambda\left(\frac{a}{c}\frac{\eta^2}{2} + a\frac{(x_c, \eta)}{c}\right) \sum_{\beta \in D_M^{c*}} \lambda\left(a\frac{\beta_c^2}{2} + b(\gamma, \beta) + bd\frac{\gamma^2}{2}\right) e_{\beta+d\gamma}.$$

Proof. According to Lemma 7.3 we can restrict the sum in Corollary 7.2 to $\beta \in D_M^{c*}$. Write such β as $x_c + c\alpha$ with $\alpha \in M^*$, and use the definition of $\frac{\beta_c^2}{2}$. Then $a(\alpha, \eta) \in \mathcal{O}$, and since the new Gauss sum is independent of β (x_c is pre-fixed) it can be factored out of the summation. This completes the proof of the corollary. \square

This result generalizes and suggests an alternative proof to Lemma 3.2 of [B3].

We note that even though $\frac{\beta_c^2}{2}$ depends only on the image of x_c in D_M , the elements $\lambda\left(\frac{a}{c}\frac{x_c^2}{2}\right)$ and the Gauss sum depend on the particular element x_c in M^* itself, because of the division by c . Their product does not depend on $x_c \in M^*$ but only on its D_M -image (since the total expression $\mathbf{r}_0(A)e_\gamma$ does not depend on x_c at all), but we shall calculate each term separately. In any case, we shall indicate exactly the choice of $x_c \in M^*$ which turns out useful for our purposes.

7.3 Determination of x_c for Lattices over \mathbb{Z}_p and \mathbb{Z}

Before we can obtain our final result, we need to specify our choice of x_c . Let M be an even lattice, and choose a Jordan decomposition for M . For any $c \neq 0$ we examine whether the Jordan component corresponding to $2^{v_2(c)}$ in the Jordan decomposition is odd (i.e., it is $(2^{v_2(c)})_I^{\varepsilon n}$ for some $t \in \mathbb{Z}/8\mathbb{Z}$) or it is even (i.e., it is $(2^{v_2(c)})_{II}^{\varepsilon n}$). In the latter case we take $x_c = 0$, noting that since M is even, this case occurs whenever c is odd. In the former case we choose an orthogonal basis for the Jordan component $(2^{v_2(c)})_I^{\varepsilon n}$, and then half of the sum of the basis vectors lies in M_2^* (since c is even). We then project it to D_{M_2} and embed it into D_M . By some abuse of notation we denote its image in all three groups by the same notation x_c . It is clear that the image in D_M coincides with x_c of [Sche] and [Str]. We prove here, in the following two lemmas, the two assertions that are given without proof on page 1494 of [Sche]:

Lemma 7.5. *The element x_c of D_M lies in D_M^{c*} .*

Proof. We have to verify that the components of x_c in D_{M_p} lie in $D_{M_p}^{c*}$ for every prime p . We already know that for odd p , 0 (which is the component of our x_c in D_{M_p}) is in $D_{M_p}^{c*}$. It remains to check for $p = 2$, and by the orthogonal direct sum property it suffices to verify that the result holds for any 2-adic Jordan component. So we take a component $q_{t/II}^{\varepsilon n}$, and distinguish the cases where $2q|c$, $2^{v_2(c)+1}|q$, and $q||c$. Recall that $x_c = 0$ except for the third case, and also in this case $x_c = 0$ unless the unimodular lattice underlying the Jordan component is odd. We now make the following observations. In the first and third cases $D_{q_{t/II}^{\varepsilon n}, c}$ is the entire Jordan component, while for the second case an element in the Jordan component lies in $D_{q_{t/II}^{\varepsilon n}, c}$ if and only if it is the $\frac{q}{2^{v_2(c)}}$ th multiple of some other vector. In any case know that the denominator of $\frac{\alpha^2}{2}$ is at most $2q$ for any element α of the dual of the Jordan component.

Now, in the first case the fact that $\frac{c}{2q} \in \mathbb{Z}_2$ implies that $c\frac{\alpha^2}{2} \in \mathbb{Z}_2$ for any such α . In the second case we find that if $\alpha \in D_{q_{t/II}^{\varepsilon n}, c}$ then $\frac{\alpha^2}{2}$ is $\frac{q^2}{2^{2v_2(c)}}$ times an element with a denominator of at most $2q$, so that multiplying this by c we obtain an integral multiple of $\frac{q}{2^{2v_2(c)+1}}$ and $c\frac{\alpha^2}{2} \in \mathbb{Z}_2$ again. In the third case with $q_{II}^{\varepsilon n}$, the denominator appearing in $\frac{\alpha^2}{2}$ is in fact at most q by the even condition, so again $c\frac{\alpha^2}{2} \in \mathbb{Z}_2$ for any α in the Jordan component. We conclude that in all these cases our choice of $x_c = 0$ indeed gives an element of $D_{q_{t/II}^{\varepsilon n}}^{c*}$. Note that the case $c = 0$ is included in the first case.

It remains to verify that for the Jordan component $q_t^{\varepsilon n}$ with $q = 2^{v_2(c)}$, the vector which is half the sum of an orthogonal basis lies in $D_{q_t^{\varepsilon n}}^{c*}$. We use again the orthogonal direct sum property in order to reduce the problem to the case $n = 1$. In this case t is odd, and the discriminant form is $(\mathbb{Z}/q\mathbb{Z})\gamma$ with $\frac{\gamma^2}{2} = \frac{t}{2q}$. Writing $c = qc_2$ with c_2 odd we find that $c\frac{\alpha^2}{2} = c_2\frac{t\alpha^2}{2}$ where $\alpha = r\gamma$. This element is just $\frac{r^2}{2}$, or equivalently $\frac{r}{2}$, in $\mathbb{Q}_2/\mathbb{Z}_2$. On the other hand, $x_c = \frac{q\gamma}{2}$ (since $q_t^{\varepsilon 1} = \mathbb{Z}_2 q\gamma$), and adding $(\alpha, x_c) = \frac{tr}{2}$ to $c\frac{\alpha^2}{2}$ gives an element of \mathbb{Z}_2 for any α . This proves the lemma. \square

Lemma 7.5 shows that x_c chosen above justifies its notation, and can be used for our purposes. Moreover, it allows us to deduce the following

Corollary 7.6. *If the Jordan decomposition we choose includes the Jordan component $(2^{v_2(c)})_{t/II}^{\varepsilon n}$ then the element x_c satisfies $\chi_2(\frac{a}{c}\frac{x_c^2}{2}) = \zeta_8^{a_2 c_2 t}$, with $t = 0$ for II or for an empty (i.e., rank 0) component.*

Proof. If we have the index II or an empty Jordan component then $x_c = 0$ and indeed $\chi_2(0) = \zeta_8^0$ (this covers the case $c = 0$, where c_2 is not defined). On the other hand, if we have an index t then x_c is half the sum of orthogonal elements v_i with $\frac{v_i^2}{q} \in \mathbb{Z}_2$ and odd for any i and $\sum_i \frac{v_i^2}{q} \equiv t \pmod{8}$. This implies that $(2x_c)^2 \equiv qt \pmod{8q\mathbb{Z}_2}$. Now, $\frac{x_c^2}{2}$ is obtained by dividing $(2x_c)^2$ by 8, and we write $c = qc_2$ with c_2 odd. Moreover, this case occurs only when c is even, hence a is odd and equals a_2 . It follows that $\frac{a}{c}\frac{x_c^2}{2} \equiv \frac{a_2 t}{8c_2} \pmod{\mathbb{Z}_2}$. Since any

odd number in \mathbb{Z}_2 is congruent to its inverse modulo 8, the proof of the corollary is now complete. \square

We remark that inside the Jordan components for which $x_c \neq 0$, x_c is determined uniquely as an element of D_M (or equivalently D_{M_2}). This can be deduced from the proof of Lemma 7.5, but it is also evident from the following argument: The entire Jordan component lies in $\Delta_{M_2, c}$, hence the corresponding Δ_M^{c*} is a coset of the trivial subgroup. In particular, it is independent (up to elements of M) of the choice of the orthogonal basis for the Jordan component. However, once Jordan components with $q \nmid c$ exist and Δ_M^{c*} contains additional elements, the element x_c does depend on the Jordan decomposition, and it is not preserved by automorphisms of the discriminant form. In fact, changing the x_c obtained by a Jordan decomposition by any element of order 2 in Δ_M^c gives an element in Δ_M^{c*} which is obtained as x_c for another Jordan decomposition.

We also note that if one wishes to lift x_c from D_M to an element of M^* then one must alter it by an appropriate element of M such that $\frac{a}{c} \frac{x_c^2}{2}$ lies in \mathbb{Z}_p for any odd p so that the formula $\mathbf{e}\left(\frac{a}{c} \frac{x_c^2}{2}\right) = \prod_p \chi_p\left(\frac{a}{c} \frac{x_c^2}{2}\right)$ holds. This is so since we assume that the p -adic parts of x_c behave as if they vanish for odd p . To avoid this technicality we shall use x_c only in M_2^* and in D_M , and write only $\chi_2\left(\frac{a}{c} \frac{x_c^2}{2}\right)$ rather than $\mathbf{e}\left(\frac{a}{c} \frac{x_c^2}{2}\right)$.

8 General Formulae for ρ_M

In this Section we derive the main result of this paper, i.e., the action of any element of $Mp_2(\mathbb{Z})$ via the representation ρ_M .

8.1 Evaluating $\rho_M(A, \varepsilon \sqrt{j(A, \tau)})$

The evaluations of Section 7 apply, in particular, for $\mathbb{F} = \mathbb{Q}_p$ and $\lambda = \chi_p$. We now take M to be an even \mathbb{Z} -lattice, and apply these results for the p -adic lattices M_p over all the primes p . Then $v = v_p$ and $q = p$, and we replace M by M_p and λ by χ_p throughout. In this way we derive an explicit formula for $\rho_M(A) = \bigotimes_p \rho_{M_p}(A)$ by substituting the expressions at our disposal, recalling that $\rho_{M_p}(A)$ differs from $\mathbf{r}_0(A)$ by the factor given in Theorem 5.2. It is expedient to recall the Weil reciprocity law, stating that $\prod_p \gamma(f_p) = \gamma(f_{\mathbb{R}}) = \zeta_8^{\text{sgn}(M)}$.

First we derive the formula from the first part of Proposition 1.6 of [Sh]. Note that the formula refers to the operators $\mathbf{r}_0(A)$ rather than to $\rho_{M_p}(A)$ (even though it is stated otherwise). Now, if $c = 0$ then $a = d = \pm 1$ and for any p , e_{γ_p} is taken to $e_{\pm \gamma_p}$ multiplied by $\chi_p\left(\pm b \frac{\gamma^2}{2}\right)$. In total we obtain $\mathbf{e}\left(\pm b \frac{\gamma^2}{2}\right) e_{\pm \gamma}$ as desired. On the other hand, if $c \neq 0$ then we take the form of $\mathbf{r}_0(A)$ given in Eq. (6). Then, $\prod_p \Delta_{M_p} = \Delta_M$ and $\prod_p p^{v_p(c)} = |c|$, and by the usual argument,

namely $\mathbf{e} = \prod_p \chi_p$ and $\sum_{\eta \in M/cM} = \prod_p \sum_{\eta \in M_p/cM_p}$, we get the expression

$$\frac{1}{|c|^{m/2} \sqrt{\Delta_M}} \sum_{\eta \in M/cM} \mathbf{e} \left(\frac{d}{c} \frac{\gamma^2}{2} - \frac{(\gamma, \delta + \eta)}{c} + \frac{a}{c} \frac{(\delta + \eta)^2}{2} \right),$$

in agreement with the expression of [Sh]. Moreover, if c is prime to the level N of M then by Lemma 2.1 multiplication by c is invertible on D_M , which reproduces Eq. (16) of [W] (again, in our form) for the group D_M .

We can now state and prove our main result. We know that ρ_{M_p} for odd p is described by Corollary 6.4 for $\mathbb{F} = \mathbb{Q}_p$ (using the Legendre symbol notation) while ρ_{M_2} is given in Corollary 6.7 (see the corresponding remarks following each corollary). With these observations, we can now state

Theorem 8.1. *For any element $(A, \varepsilon \sqrt{j(A, \tau)}) \in Mp_2(\mathbb{Z})$ (with $\sqrt{j(A, \tau)}$ having its argument in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ as usual) we have that $\rho_M(A, \varepsilon \sqrt{j(A, \tau)}) e_\gamma$ is*

$$\prod_p \xi_p \cdot \frac{\sqrt{\Delta_{M,c}}}{\sqrt{\Delta_M}} \sum_{\beta \in D_M^*} \mathbf{e} \left(a \frac{\beta_c^2}{2} + b(\gamma, \beta) + bd \frac{\gamma^2}{2} \right) e_{\beta+d\gamma},$$

with the root of unity ξ_p defined to be $(\frac{a_p}{\Delta_{M_p}}) \prod_{q|c} \overline{\gamma(q^{\varepsilon n}(a_p c))}$ for odd p and

$$\varepsilon^m \left(\frac{a}{c_2} \right)^m (-1)^{m\varepsilon(a_2)\varepsilon(c_2)} \left(\frac{2^{v_2(c)}}{a_2} \right)^m \left(\frac{\Delta_{M_2}}{a_2} \right) \gamma(f_2)^{a_2-1} \prod_{q|c} \overline{\gamma(q_{t/II}^{\varepsilon n}(a_2 c))}$$

for $p = 2$. Here we take the convention in which for $c = 0$ the odd number c_2 is positive.

Proof. Note that for $c \neq 0$ our choice of x_c agrees with the choice of x_c in Theorem 4.3. Hence we write $\mathbf{r}_0(A) e_{\gamma_p}$ as in Corollary 7.4 and evaluate the Gauss sum by Theorem 4.3, which yields

$$\delta_p \frac{\sqrt{\Delta_{M_p,c}}}{\sqrt{\Delta_{M_p}}} \sum_{\beta_p \in D_{M_p}^*} \chi_p \left(a \frac{\beta_{p,c}^2}{2} + b(\gamma_p, \beta_p) + bd \frac{\gamma_p^2}{2} \right) e_{\beta_p+d\gamma_p}$$

(note that $p^{mv_p(c)/2}$ is canceled). Here δ_p is the factor $\prod_{pq|c} \gamma(q_{t/II}^{\varepsilon n}(a_p c))$ appearing in Theorem 4.3, while for $p = 2$ we have an additional factor of $\zeta_8^{a_2 c_2 t_c}$, where t_c is the index t appearing in the block $q_{t/II}^{\varepsilon n}$ for $q = 2^{v_2(c)}$. This extra factor is the evaluation of $\chi_2(\frac{a}{c} \frac{x^2}{2})$ given in Corollary 7.6. Note that for every odd prime p satisfying the conditions of Lemma 2.1 and not dividing c , the representation is 1-dimensional, the real number and the coefficient in the sum are 1, and δ_p is also 1 since the product is empty. Hence it is allowed to take the tensor product over all primes p because it is essentially finite. The equalities $\prod_p \Delta_{M_p} = \Delta_M$ and $\prod_p \Delta_{M_p,c} = \Delta_{M,c}$ and the local-to-global nature of Δ_M^* show that the tensor product yields the asserted summation and real

constant. It remains to verify that multiplying δ_p by the metaplectic roots of unity differentiating ρ_{M_p} from the corresponding $\mathbf{r}_0(A)$ yields the asserted ξ_p .

Now, Corollaries 6.4 (with the Hilbert symbol for \mathbb{Q}_p replaced by the Legendre symbol as noted there) and 6.7 imply that the desired root of unity ξ_p is $\delta_p \left(\frac{a_p}{p^{v_p(c)}} \right)^m \overline{\gamma(cf_p)}$ for odd p and $\delta_2 \varepsilon^m \left(\frac{a}{c_2} \right)^m \overline{\gamma(cf_2)}$ for $p = 2$. First we consider the case where $a \neq 0$, which allows us to write $\overline{\gamma(cf_2)}$ as $\gamma(cf_2)^{a_2-1} \overline{\gamma(cf_2)^{a_2}}$ and use Lemma 4.2. Since the cardinality of the discriminant of $M_p(c)$ is $p^{mv_p(c)} \Delta_{M_p}$, Lemma 4.2 allows us to replace $\overline{\gamma(cf_p)}$ for odd p and $\overline{\gamma(cf_2)^{a_2}}$ for $p = 2$ by $\left(\frac{a_p}{p^{mv_p(c)} \Delta_{M_p}} \right) \overline{\gamma(a_p cf_p)}$ and $\left(\frac{2^{mv_2(c)} \Delta_{M_2}}{a_2} \right) \overline{\gamma(a_2 cf_2)}$ respectively. For odd p we can now cancel the two $\left(\frac{a_p}{p^{v_p(c)}} \right)^m$ factors. On the other hand, for $p = 2$ the remark about powers of 2 preceding Lemma 4.2 shows that we can replace $\gamma(cf_2)^{a_2-1}$ by $\gamma(c_2 f_2)^{a_2-1}$, as the Weil indices differ by a sign and $a_2 - 1$ is even. Moreover, we can apply Lemma 4.2 again to write $\gamma(c_2 f_2)^{a_2-1}$ as $\gamma(f_2)^{a_2-1} (-1)^{m\varepsilon(a_2)\varepsilon(c_2)}$ (after writing c_2 as $1 + (c_2 - 1)$). This follows since the even power $a_2 - 1$ allows us to ignore the associated Legendre symbol, $4|(c_2 - 1)(a_2 - 1)$ (with the quotient being the product of the ε s in \mathbb{F}_2), and $\gamma(f_2)^4 = (-1)^m$ since the oddity and rank of M always have the same parity. In any case, when we decompose $\overline{\gamma(a_p cf_p)}$ as the product of the Weil indices of the Jordan components of M_p (multiplied by $a_p c$) we see that it cancels with δ_p , leaving the asserted product over q not dividing c . It only remains to check that the part with $q|c$ cancels as well. Here we have $\gamma((p^{2v_p(c)})^{\varepsilon_n}_{/t_c/II}(a_p c_p))$ for any p together with the element $\zeta_8^{a_2 c_2 t_c}$ for $p = 2$. Since the power of p is even in this component, Proposition 4.1 shows that this complex conjugate Weil index is 1 for odd p and cancels with $\zeta_8^{a_2 c_2 t_c}$ for $p = 2$. Thus, ξ_p has the asserted value if $c \neq 0$ and $a \neq 0$ which completes the proof for this case.

For $a = 0$ the number a_p is not well-defined. However, $\delta_p = 1$ for all p (since $c = \pm 1$), and our expansion process gives the same value for every choice of $a_p \in \mathbb{Z}_p^*$. Therefore the asserted formula is independent of the choice of a_p and gives the desired value for any a_p . In fact, in this case undoing the expansion process yields the simpler formulae $\xi_p = \overline{\gamma(cf_p)}$ for odd p and $\xi_2 = \varepsilon^m \overline{\gamma(cf_2)}$ (since also $\left(\frac{a}{c_2} \right) = 1$ then). This simpler formula for ξ_p (with $\left(\frac{a}{c_2} \right)$ added to ξ_2) is more generally valid whenever c is not divisible by p (by the same argument). Nonetheless, we use the asserted formula since it applies for every possible case.

We now explain why the same formula holds also for $c = 0$. We have seen that $D_M^{0*} = \{0\}$ with $\frac{\beta_0^2}{2} = 0$, $\Delta_{M,0} = \Delta_M$, and $a = d = \pm 1$, so that the asserted formula becomes $\prod_p \xi_p \cdot \mathbf{e}(\pm b \frac{\gamma^2}{2}) e_{\pm \gamma}$. It remains to verify that $\xi_p = \gamma(a f_p) \overline{\gamma(f_p)}$ (with the additional factor ε^m for $p = 2$). But for $c = 0$ there is no q not dividing c , which leaves ξ_p for odd p with the value $\left(\frac{a_p}{\Delta_{M_p}} \right)$. Since $a_p = a = \pm 1$ we have $\xi_p = \left(\frac{\pm 1}{\Delta_{M_p}} \right)$. This settles the case $a = d = +1$, and the $a = d = -1$ case follows from the remark after Lemma 4.2. For $p = 2$ we also note that $v_2(a) = 0$ (hence $a_2 = a$) and of course $v_2(c_2) = 0$. Now, since the Legendre symbol over $a_2 = \pm 1$ always gives 1 we find that Eq. (5) for a

and c_2 reduces ξ_2 to $\varepsilon^m \gamma(f_2)^{a-1} (c_2, -1)_{\mathbb{R}}^m$. By our convention $c_2 > 0$, hence the Hilbert symbol is 1 and ξ_2 has the desired value as well. This completes the proof of the theorem. \square

We remark that we already know from Lemma 2.1 that for any p not dividing Δ_M we must have $\xi_p = 1$. Indeed, take p to be such a prime. Then the Legendre symbol over $\Delta_{M_p} = 1$ (or under $\Delta_{M_2} = 1$ if $p = 2$) equals 1, and there exists no q not dividing c since we only have $q = 1$. Moreover, if $p = 2$ does not divide Δ_M then $\gamma(f_2) = 1$ as well, and the fact that m must be even in this case completes the verification. Note that unlike the assertion $\xi_p = 1$ for p not dividing Δ_M , similar assertion does not necessarily hold for the δ_p appearing in the proof of Theorem 8.1, since δ_p may be non-trivial for such p in case $p|c$.

It may be more convenient to rephrase Theorem 8.1 without referring to a specific choice of the branch of $\sqrt{j(A, \tau)}$. For this purpose we can replace, for $c \neq 0$, the metaplectic sign ε by the chosen sign of the real part of the square root of $j(A, \tau)$. As for $c = 0$, $\sqrt{j(A, \tau)}$ is a constant number δ which satisfies $\delta^2 = a = d$, and more explicitly $\delta = \varepsilon$ if $a = d = 1$ and $\delta = -i\varepsilon$ if $a = d = -1$. We claim that the total coefficient appearing in $\rho_M(A, \delta)$ in this case is just $\delta^{-\text{sgn}(M)}$. Indeed, for $a = d = 1$ all we have is ε^m , which is the desired value since m and $\text{sgn}(M)$ have the same parity. For $a = d = -1$ we use the Weil Reciprocity Law to replace the total product $\prod_p \gamma(f_p)^2$ by $i^{\text{sgn}(M)}$, and again the power of ε yields the asserted expression. Since choosing the branch of $\sqrt{j(A, \tau)}$ with argument in $(-\frac{\pi}{2}, \frac{\pi}{2}]$ gives $\delta = \varepsilon_a$, this agrees with the result of [Str] for this case.

8.2 Congruence Subgroups of Level N

Let us denote the exponent of D_M by \tilde{N} . Equivalently, \tilde{N} is the minimal number by which multiplying (γ, δ) with γ and δ in M^* always gives an element of \mathbb{Z} . Using a Jordan decomposition of M_p for any p shows that \tilde{N} is the least common multiple of all the numbers q appearing in the Jordan components, while Δ_M is clearly the product of all of them. Moreover, the level of M is the least common multiple of the levels of the Jordan components. The level of q^{ε^n} (with odd p) is q , the level of $q_{II}^{\varepsilon^n}$ is also q , and the level of $q_t^{\varepsilon^n}$ is $2q$, and N is the least common multiple of all these numbers. Hence N equals \tilde{N} if the Jordan component corresponding to $2^{v_2(\tilde{N})}$ comes with index II and equals $2\tilde{N}$ if it comes with some other index t . This implies Corollary 3.3: The components $q_{II}^{\varepsilon^n}$ come only with even ranks, so that if $rk(M)$ is odd then M_2 must contain some Jordan component of the form $q_t^{\varepsilon^n}$ of level $2q$. Since M is assumed to be an even lattice then this cannot occur with $q = 1 = 2^0$, giving a component whose level is divisible by 4. In addition, this argument proves the part $(i) \Leftrightarrow (ii)$ in Lemma 2.1: Clearly \tilde{N} and Δ_M must have the same prime divisors by our description, so that if $N = \tilde{N}$ we are done. The other option is $N = 2\tilde{N}$, and then only the prime requires additional consideration. However, when this situation occurs there exists a Jordan component of the form $q_t^{\varepsilon^n}$ with $q = 2^{v_2(\tilde{N})}$. Since this

cannot happen if $v_2(\tilde{N}) = 0$ we deduce that $2|\tilde{N}$ and indeed N and \tilde{N} (hence N and Δ_M) have the same prime divisors. Moreover, the description of \tilde{N} , N , and Δ_M imply that $\tilde{N}|\Delta_M$ and $N|2\Delta_M$.

We recall that $\Gamma_0(N)$ is the group of elements in $SL_2(\mathbb{Z})$ (with the usual entries) such that $N|c$, and the group $\Gamma_0^0(N)$ is defined to be the subgroup of $\Gamma_0(N)$ consisting of those elements in which b is also divisible by N . The group $\Gamma(N)$ of matrices congruent to the identity modulo N is normal and can be seen as the group of elements of $\Gamma_0^0(N)$ in which $a \equiv d \equiv 1 \pmod{N}$. Define Γ , for N and \tilde{N} given as above, to be the subgroup of $\Gamma_0^0(N)$ in which $a \equiv d \equiv 1 \pmod{\tilde{N}}$. It contains $\Gamma(N)$ as a subgroup of index $\frac{N}{\tilde{N}}$, and it is normal in any case (as can be easily checked by conjugation modulo $\Gamma(N)$).

Now assume that A is a matrix in $\Gamma_0(\tilde{N})$. Then we have $\Delta_M^c = \{0\}$, i.e., $\Delta_M^{c*} = \{x_c\}$ and $\frac{\beta_c}{2} = 0$ for $\beta = x_c$, and also $\Delta_{M,c} = \Delta_M$ (all these assertions are equivalent to one another as well as to the assertion $\tilde{N}|c$). Moreover, every q appearing in a Jordan component of M must divide c , since we have seen that it divides N and $N|c$. According to Lemma 2.1, if p does not divide N then $\Delta_{M_p} = 1$. For any other p we have $p|c$, hence p does not divide a since a and c are coprime. If N is odd then m is even by Corollary 3.3, hence every ± 1 raised to the m th power is 1 and $\gamma(f_2) = 1$. Therefore we can replace a_p by a anywhere. Since $\prod_{p \neq 2} \Delta_{M_p} = \Delta_{M,2}$ Theorem 8.1 implies, for any $\gamma \in D_M$, that $\rho_M(A, \varepsilon \sqrt{j(A, \tau)})e_\gamma$ equals

$$\varepsilon^m \left(\frac{a}{c_2} \right)^m (-1)^{m\varepsilon(a)\varepsilon(c_2)} \left(\frac{2^{v_2(c)}}{a} \right)^m \left(\frac{\Delta_{M_2}}{a} \right) \left(\frac{a}{\Delta_{M,2}} \right) \gamma(f_2)^{a-1} \mathbf{e} \left(b(\gamma, x_c) + bd \frac{\gamma^2}{2} \right) e_{x_c + d\gamma}.$$

Moreover, if we assume that $N|c$ then $x_c = 0$ (again, these are equivalent assertions if $\tilde{N}|c$). Hence if we define $\varphi(A, \varepsilon \sqrt{j(A, \tau)})$ for $A \in \Gamma_0(N)$ to be

$$\varepsilon^m \left(\frac{a}{c_2} \right)^m (-1)^{m\varepsilon(a)\varepsilon(c_2)} \left(\frac{2^{v_2(c)}}{a} \right)^m \left(\frac{\Delta_{M_2}}{a} \right) \left(\frac{a}{\Delta_{M,2}} \right) \gamma(f_2)^{a-1}$$

then we obtain

$$\rho_M(A, \varepsilon \sqrt{j(A, \tau)})e_\gamma = \varphi(A, \varepsilon \sqrt{j(A, \tau)}) \mathbf{e} \left(bd \frac{\gamma^2}{2} \right) e_{d\gamma}.$$

φ is a character of the inverse image of $\Gamma_0(N)$ in $Mp_2(\mathbb{Z})$, since the action on e_0 is just multiplication by $\varphi(A, \varepsilon \sqrt{j(A, \tau)})$. When $A \in \Gamma_0^0(N)$, then the action of $(A, \varepsilon \sqrt{j(A, \tau)})$ via ρ_M is multiplication of the index γ by d and multiplying the vector thus obtained by the scalar $\varphi(A, \varepsilon \sqrt{j(A, \tau)})$. These assertions are interesting on their own right, and are also useful here for proving the following

Proposition 8.2. *The kernel of ρ_M is a (normal) subgroup of $Mp_2(\mathbb{Z})$ which almost always lies over Γ defined above, except for a few cases in which it lies over $\Gamma(N)$. These cases are (i) $2||\tilde{N}$ and $\gamma(f_2)^2 \neq 1$ (which always holds for odd m), and (ii) m is even, $4||\tilde{N}$, and $v_2(\Delta_M)$ is odd. This kernel is a double*

cover of Γ or $\Gamma(N)$ respectively if m is even, and is a (normal) lift of this group if m is odd.

Proof. Any element $(A, \varepsilon \sqrt{j(A, \tau)})$ of $\ker \rho_M$ must lie above an element of $\Gamma_0(N)$, since $\Delta_M^* = \{0\}$, i.e., $\Delta_M^c = \{0\}$ and $x_c = 0$. In addition, examining the action on e_0 shows that it must lie in $\ker \varphi$. Moreover, a trivial action on e_γ implies $bd\frac{\gamma^2}{2} \in \mathbb{Z}$ and $d\gamma = \gamma$ for any γ . The first property implies that $N|bd$, and since d is prime to N (as it is to c and $N|c$) it follows that $A \in \Gamma_0^0(N)$. The second property shows that $\tilde{N}|d-1$, and since $N|c$ we have $\tilde{N}|a-1$ as well. Hence every element of $\ker \rho_M$ lies over Γ . Since $\ker \rho_M$ is now seen to consist of those elements lying over Γ which are in $\ker \varphi$, we now turn to characterize the latter kernel.

We now claim that the image of φ on the inverse image of $\Gamma(N)$ in $Mp_2(\mathbb{Z})$ is $\{(\pm 1)^m\}$. First we establish that the power of $\gamma(f_2)$ gives a ± 1 in this case: If $4|N$ then $4|a-1$ and $\gamma(f_2)^{a-1}$ is a power of $\gamma(f_2)^4 = (-1)^m$. On the other hand, if 4 does not divide N then the only Jordan components which can appear in M_2 (being even and of level indivisible by 4) are $1_{II}^{\varepsilon n}$ and $2_{II}^{\varepsilon n}$, and their Weil indices are signs. The next step is showing that $(\frac{a}{\Delta_{M_p}}) = 1$ for any odd prime p if A lies over Γ . Indeed, for p not dividing N Lemma 2.1 implies $\Delta_{M_p} = 1$, while for $p|N$ it follows since $a \equiv 1 \pmod{N}$. We continue by proving that $\gamma(f_2)^{a-1}$ and $(\frac{\Delta_{M_2}}{a})$ also equal 1 on $\Gamma(N)$. If N is odd then the 2-adic lattice M_2 is unimodular, hence $\Delta_{M_2} = 1$ and the Jordan decomposition of M_2 must then consist of one component of the form $1_{II}^{\varepsilon n}$ (with $n = m$). Since such a component has Weil index 1, the assertion holds for odd N . If $2||N$ then the additional component of the form $2_{II}^{\varepsilon n}$ contributes a square to Δ_{M_2} , and since its Weil index is ε and a is odd, the assertion holds in this case as well. If $4||N$ then the Jordan component of the form $4_{II}^{\varepsilon n}$ has no effect on both expressions, and it remains to consider the effect of the Jordan component $2_t^{\varepsilon n}$. The contribution to $\gamma(f_2)^{a-1}$ is $(-1)^{n\frac{a-1}{4}}$ since $t \equiv n \pmod{2}$, and the Legendre symbol is multiplied by $(\frac{2}{a})^n$. Now, Eq. (1) gives $(\frac{2}{a}) = (-1)^{\frac{a-1}{4}}$ for $a \in 1 + 4\mathbb{Z}_2$ (since $\varepsilon_a = 1$ and $\zeta_8^4 = -1$), which confirms the claim for this case. Finally, if $8|N$ then the Weil index is taken to a power which is divisible by 8 and the Legendre symbol is a power of 2 over an element on $1 + 8\mathbb{Z}_2$. Hence both multipliers are 1 and the assertion is clear. This implies that on $\varphi(A, \varepsilon \sqrt{j(A, \tau)})$ lying over $\Gamma(N)$ the image of the character φ reduces to $\varepsilon^m (\frac{a}{c_2})^m (-1)^{m\varepsilon(a)\varepsilon(c_2)} (\frac{2^{v_2(c)}}{a})^m$ (which is independent of the lattice), and in particular for even m this element of $Mp_2(\mathbb{Z})$ lies in $\ker \varphi$ and for odd m precisely one choice of ε gives an element of $\ker \varphi$.

Summarizing, we find that $\ker \rho_M$ lies over a group between Γ and $\Gamma(N)$, with the asserted behavior with respect to the parity of m . This completes the proof of the proposition for the case where $N = \tilde{N}$. If $N = 2\tilde{N}$ then since $[\Gamma : \Gamma(N)] = 2$, $\ker \rho_M$ lies either over Γ or over $\Gamma(N)$. In order to determine $\ker \rho_M$ completely, consider an element $\varphi(A, \varepsilon \sqrt{j(A, \tau)})$ lying over Γ but not over $\Gamma(N)$. Hence $a-1$ and $d-1$ are divisible by \tilde{N} but not by N . We need to verify that up to the choice of ε in the case of odd m , this element lies in $\ker \varphi$ unless we are in one of the asserted cases (i) and (ii). The argument

showing that $(\frac{a}{\Delta_{M,2}}) = 1$ holds for Γ as well as for $\Gamma(N)$, since only odd primes are considered there. Moreover, all the elements with power m equal 1 for even m and only determine the choice of ε if m is odd, hence have no effect on the question whether the element we consider (up to ε) lies in $\ker \rho_M$ or not. Therefore only the part $(\frac{\Delta_{M_2}}{a})\gamma(f_2)^{a-1}$ requires our attention. The same argument as above shows that if $8|\tilde{N}$ then the latter expression equals 1, hence in this case $\ker \rho_M$ lies over Γ . Moreover, the case $N = 2\tilde{N}$ can occur only if \tilde{N} is also even, hence we only have to check the cases $2||\tilde{N}$ and $4||\tilde{N}$. Now, if $2||\tilde{N}$ then we have seen that the only Jordan component which has non-trivial contribution to $(\frac{\Delta_{M_2}}{a})\gamma(f_2)^{a-1}$ is $2_t^{\varepsilon n}$, and that in this case m , n , and t all have the same parity. Since $a - 1$ is divisible by \tilde{N} but not by N we find that $a - 1$ is even but $\frac{a-1}{2}$ is odd, so that $\zeta_8^{a-1} = \pm i$ and $\gamma(f_2)^{a-1} = (\pm i)^t$. Since $(\frac{\Delta_{M_2}}{a}) = (\pm 1)^n = (\pm 1)^m$ we see that for odd m (and t) the φ -image in question lies in $\pm i$ while for even m it gives just $\gamma(f_2)^2$ (since $\gamma(f_2)^4 = 1$). It follows that the only possibility for $\varphi(A, \varepsilon\sqrt{j(A, \tau)})$ to be in $\ker \rho_M$ is if m is even and $\gamma(f_2)^2 = 1$, and case (i) is proved. If $4||\tilde{N}$, i.e., $a - 1$ is divisible by 4 and not by 8, then we know that $\gamma(f_2)^{a-1} = (-1)^m$. Moreover, since $a \equiv 5 \pmod{8}$ we find that $(\frac{\Delta_{M_2}}{a}) = (-1)^{v_2(\Delta_M)}$. Therefore for odd m $\varphi(A, \varepsilon\sqrt{j(A, \tau)})$ is in ± 1 and the right choice of ε gives an element in $\ker \rho_M$. On the other hand, for even m we have $\varphi(A, \varepsilon\sqrt{j(A, \tau)}) = (\frac{\Delta_{M_2}}{a}) = (-1)^{v_2(\Delta_M)}$. Hence $A, \varepsilon\sqrt{j(A, \tau)}$ lies in $\ker \rho_M$ if and only if $v_2(\Delta_M)$ is even, which proves case (ii).

This completes the proof of the proposition. \square

The result of Proposition 8.2 contains that of Theorem 3.2, and gives conditions under which Theorem 3.2 describes all of $\ker \rho_M$. When these conditions are not satisfied, Theorem 3.2 describes a subgroup of index 2 of this kernel. Proposition 8.2 also implies that the group Γ is normal in $SL_2(\mathbb{Z})$ also when $N = 2\tilde{N}$ (a fact which can be easily verified directly, working modulo $\Gamma(N)$), and if $4||\tilde{N}$ then it can be lifted to a subgroup of $Mp_2(\mathbb{Z})$ which is normal as well. In fact, the proof of Proposition 8.2 shows that the restriction of φ to $\Gamma(N)$ is independent of the lattice, so that the (normal) lift of $\Gamma(N)$ into $Mp_2(\mathbb{Z})$ defined by φ for odd m is just the restriction of a given lift of $\Gamma(4)$. The same assertion holds for the lift of Γ in the case where $N = 2\tilde{N}$ and $8|\tilde{N}$ (with odd m), but in the case where $4||\tilde{N}$ the lift is multiplied by the character χ_8 of [B3] taken to the power $v_2(\Delta_M) + 1$. Explicitly, it follows from the proof of Proposition 8.2 that if m is odd then for any element in $\ker \rho_M$ we must have $a \equiv 1 \pmod{4}$. Hence $(-1)^{m\varepsilon(a)\varepsilon(c_2)} = 1$ and this lift is the one described in Corollary 6.8. Applying this to a lattice of odd rank and level 4 (e.g., $M = \mathbb{Z}x$ with $x^2 = 2$) provides an alternative proof of Corollary 6.8. In addition, by Lemma 2.1 we have $\rho_M = \rho_{M_2}$ in the decomposition of Section 2 in this case. Therefore, applying the map i from Theorem 6.6 to this kernel yields the intersection of $\ker \rho_{M_2}$ with the inverse image of $SL_2(\mathbb{Z})$ in $Mp_2(\mathbb{Z}_2)$. Since the inverse image of $SL_2(\mathbb{Z})$ is dense in $Mp_2(\mathbb{Z}_2)$ (and ρ_{M_2} is continuous), the closure of $i(\ker \rho_M)$ is $\ker \rho_{M_2}$. Unwinding the definitions, we find that $\ker \rho_{M_2}$ in this case is pre-

cisely $\iota(\Gamma(4, \mathbb{Z}_2))$, for ι the map from Theorem 6.3. This provides an alternative proof of Corollary 6.5 for $\mathbb{F} = \mathbb{Q}_2$, as suggested above. More generally, the restriction of φ to elements lying over $\Gamma_0(N) \cap \Gamma(4)$ (or over $\Gamma_0(N) \cap \Gamma_1(4)$) for odd m yields a lift into $Mp_2(\mathbb{Z})$ which is the one described in Corollary 6.8 but multiplied by the character $\chi_{2\Delta_M}$ in the notation of [B3]. This is related to the maps defined in Theorem 5.4 of [B3] and Lemma 5.7 of [Str].

8.3 Comparison with Known Formulae

We have seen that for an element in $Mp_2(\mathbb{Z})$ in which the c -entry vanishes, our result for the ρ_M -image coincides with that of [Str] and [Sche]. Let us now compare our results to those of [Str] and [Sche] also for the case where $c \neq 0$. We recall that the factors of [Sche] must be conjugated, since the complex conjugate representation is employed in this reference. We need to consider only the roots of unity ξ_p (completing with ξ_0). We also note that ξ_0 of [Str] can be written as $i^{-\text{sgn}(M)}$ times the m th power of the additional expression (since the latter is defined to be 1 for even m and a term equalling ± 1 for odd m). This additional expression does not appear in [Sche] since only the case of even signature (or rank) is treated there. The product appearing in ξ_p is non-empty only if $p|c$ (so that we can replace a_p by a there), and if q does not divide c then $q_c = \gcd\{c, q\}$ is $p^{v_p(c)}$ and c/q_c is just c_p . Moreover, we write any Jordan component $q_{/t/II}^{\varepsilon n}(ac)$ as $(p^{v_p(c)}q)_{/t/II}^{\varepsilon n}(ac_p)$, and since the Weil index of such a Jordan component appears in ξ_p only if q does not divide c we can replace this Weil index by that of $(\frac{q}{p^{v_p(c)}})_{/t/II}^{\varepsilon n}(ac_p)$ (as these Weil indices coincide).

We start with odd p , where we decompose $(\frac{a_p}{\Delta_{Mp}})$ as $\prod_q (\frac{a_p}{q^n})$ and use Lemma 4.2. This gives us for $p|c$ the (conjugated) expression of [Sche] but multiplied by $\prod_q (\frac{-1}{q^n}) = (\frac{-1}{\Delta_{Mp}})$ (since $a = a_p$). If p does not divide c then $q_c = 1$ and the Legendre symbols with a_p cancel, and in total the symbols with $c_p = c$ give the (conjugated) corresponding formula of [Sche] multiplied by $(\frac{-1}{\Delta_{Mp}})$ (this is easily seen if we write our ξ_p as $\overline{\gamma(cf)}$). Now, we have seen that the (conjugated) expression of [Sche] for the case $p|c$ holds also if p does not divide c , if a is replaced by a_p . We further observe that Lemma 4.2 with the multiplier -1 takes the latter formula to the expression appearing in [Str]: One must only notice that for $q|c$ we have $q_c = q$ and for the other q the Legendre symbols with $-a$ combine again to give $(\frac{-a}{q_c^n})$. This observation also holds for c prime to p since then $q_c = 1$ for every q . We remark that unlike [Sche] and [Str] we include the case $q = 1$, but this does not introduce any additional terms since $q = 1$ always divides c and the Legendre symbol over it is always 1. In total, our result differs by $(\frac{-1}{\Delta_{Mp}})$ from the expressions in both other references, and

according to the remark following Lemma 4.2, this factor is $\overline{\gamma(f_p)}^2$.

We now turn to $p = 2$, where we recall that $\gamma = \zeta_8^{\text{oddy}}$. Hence inserting a coefficient in front of the oddity gives powers of the Weil index. We start with the case of even c , where $a_2 = a$, and apply Lemma 4.2 to our Weil indices in which q not dividing c . We observe that the expression containing the oddity

with q not dividing c in [Str] can be written as $\gamma\left(\left(\frac{a}{q_c}\right)_{t/II}^{\varepsilon n}\right)^{-ac_p}$, and the product of the (conjugated) expressions with the oddity and with γ_2 in [Sche] for the same q yields the same value. Therefore we obtain exactly that part of the expression in [Sche]. For $q|c$ we have $q_c = q$ and for other q the Legendre symbols over q_c and over q differ by the Legendre symbol over the quotient q/q_c (as in a Legendre symbol in which the numerator is a power of 2, the sign of the denominator makes no difference). Hence the corresponding parts in our results agree with those of [Sche] and [Str]. The remaining product of Legendre symbols reduces to our $\left(\frac{\Delta_{M_2}}{a_2}\right)$. Even though the (conjugated) remaining expression in [Sche] gives $\gamma(f_2)^{a+1}$, it can be replaced by its conjugate $\gamma(f_2)^{-1-a}$ (appearing in [Str]). This holds because the total oddity is even in the cases considered in [Sche] and the power $a+1$ is even, so that this number is real (i.e., just ± 1). Moreover, our $(-1)^{m\varepsilon(a_2)\varepsilon(c_2)}$ cancels with the effective $\mathbf{e}\left(m\frac{(c_2+1)(a+1)}{8}\right)$ of [Str] to give $(-1)^{m\varepsilon(c_2)-m(a+1)/2}$. Now, [Str] assumes that $c > 0$ and that we take the principal $\sqrt{j(A, \tau)}$, so that $\varepsilon = 1$ and we are in the case where our convention for the Legendre symbol yields the same value as that of [Str] (up to the even denominator). This means that the (effective) $\left(\frac{-a}{c}\right)^m$ of [Str] cancels with our $\left(\frac{a}{c_2}\right)^m \left(\frac{2^{v_2(c)}}{a_2}\right)^m$ to give just $\left(\frac{-1}{c_2}\right)^m = (-1)^{m\varepsilon(c_2)}$. Combining all this we see that our expression multiplies that of [Str] and [Sche] by $\gamma(f_2)^{2a}(-1)^{-m(a+1)/2}$. The latter multiplier is $\gamma(f_2)^{-2(a+1)}$ (recall that $\gamma(f_2)^4 = (-1)^m$), and the ratio reduces to $\overline{\gamma(f_2)}^2$.

Turning now to the case where c is odd, the product of the elements in the the (conjugated) expression involving the oddity in [Sche] becomes $\overline{\gamma(f_2)^{c+1}}$. The same reasoning as with a in the previous paragraph shows that this expression can be written without the complex conjugation. Thus, this term combines with the terms involving γ_2 (which multiply to just $\overline{\gamma(f_2)}$) to give the same $\gamma(f_2)^c$ of [Str]. Since Legendre symbols under powers of 2 are insensitive to signs, the expressions of [Str] and [Sche] coincide. Now, we write our ξ_2 for this case in the form not including a_2 . The term $\left(\frac{\Delta_{M_2}}{c}\right)$ which comes from applying Lemma 4.2 cancels with their Legendre symbols. Again our $\left(\frac{a}{c_2}\right)^m$ cancels with the $\left(\frac{-a}{c}\right)^m$ of [Str] to give $\left(\frac{-1}{c}\right)^m = (-1)^{m\varepsilon(c)}$ (since $c > 0$ and $v_2(c) = 0$), so that in total our expression multiplies that of [Str] and [Sche] by $\overline{\gamma(f_2)^{2c}}(-1)^{m(c-1)/2}$. The same considerations we applied at the end of the previous paragraph show that the power of -1 is $\gamma(f_2)^{2c-2}$, so that the total ratio is, once again, $\overline{\gamma(f_2)}^2$.

We have found that our ξ_p multiplies ξ_p of [Str] and [Sche] by $\overline{\gamma(f_p)^2}$ for every prime p . However, these authors have an additional multiplier of $i^{-sgn(M)}$ in their ξ_0 . According to the Weil Reciprocity Law this root of unity equals $\prod_p \overline{\gamma(f_p)^2}$, and we conclude that all the results are consistent. Moreover, if $c = 1$ and $ad = 0$ (or just $a = 0$) then we can write our ξ_p as $\overline{\gamma(f_p)}$ for any p (including $p = 2$ since $\left(\frac{a}{c_2}\right) = 1$ as $c_2 = 1$ and we look only at $\varepsilon = 1$). Multiplying their product $\zeta_8^{sgn(M)}$ by $i^{-sgn(M)}$ from ξ_0 yields the $\zeta_8^{-sgn(M)}$ asserted in [Str]. Finally, to transform a matrix with $c > 0$ with the principal $\sqrt{j(A, \tau)}$ to its

negative, with $c < 0$, but also with the principal square root, we have to multiply by $Z^{-1} = (-I, -i)$, whose ρ_M -image contains the root of unity $i^{\text{sgn}(M)}$. The consistency check is now complete. We just add the following remark about generalizing of the last assertion to any A . In order to take A to $-A$ while keeping both $\sqrt{j(A, \tau)}$ and $\sqrt{j(-A, \tau)}$ with argument in $(-\frac{\pi}{2}, \frac{\pi}{2}]$, one has to multiply A by $(-I, i)$ if $c < 0$ or if $c = 0$ and $d > 0$, and by $(-I, -i)$ if $c > 0$ or if $c = 0$ and $d < 0$. The verification is simple and straightforward, and this agrees with Lemma 5.4 of [Str].

Next, we compare the results for elements in $Mp_2(\mathbb{Z})$ lying over $\Gamma_0(N)$. Observe that for even m the character φ is the one denoted χ_D in [Sche], hence the paragraph preceding Proposition 8.2 proves Propositions 4.2, 4.4, and 4.5, as well as the first assertion in Proposition 4.8, of [Sche]. Indeed, replacing a by d in the definition of φ gives the same value (using the usual argument for the Legendre symbols over c_2 and $\Delta_{M,2}$, and checking case by case following the proof of Proposition 8.2), which confirms the relevant assertion in Proposition 4.4 of [Sche]. On the other hand, the assertion about c being coprime to N just before Theorem 8.1 proves, together with Lemma 4.2 and the Weil Reciprocity Law, the second assertion of Proposition 4.8 there. In fact, these tools also give Lemma 5.6 of [Str]. Furthermore, Eq. (5) shows that for $c > 0$ we can replace the terms with power m in the definition of φ by $\varepsilon^m \left(\frac{c}{d_2}\right)^m \left(\frac{2^{v_2(d)}}{c_2}\right)^m$, so that we indeed obtain $\chi_{\mathcal{Q}}(\mathbf{A})$ of [Str]. Thus, the same arguments also prove Lemmas 5.12 and 5.13 of that reference. We note further that extending the lift from Corollary 6.8 to $\Gamma_1(4)$ gives the kernel of χ_θ from [B3]: Elements in that kernel must lie over $\Gamma_1(4)$ to avoid the power of i , and since the Legendre symbol in [B3] is written here as $\left(\frac{c}{d}\right)(-1)^{\sigma(c)\sigma(d)}$ (note the difference of conventions for the Legendre symbol!), the statement follows from Eq. (5) and the fact that $d \equiv 1 \pmod{4}$. Thus, Lemma 5.7 of [Str] shows that the argument above also implies Theorem 5.4 of [B3]. One can also compare these results to part (ii) of Proposition 1.6 of [Sh], using Eq. (1), the Weil Reciprocity Law, the quadratic reciprocity law, and taking into account the convention difference in the Legendre symbols and the fact that the determinant denoted there by D equals $(-1)^{\frac{m-\text{sgn}(M)}{2}} \Delta_M$ (if $M_{\mathbb{R}}$ is of signature (p, q) then $q = \frac{m-\text{sgn}(M)}{2}$). At this point we find a difference between our results and those of [Sh]: For example, if m is odd then a must be odd since $4|N|c$ by Corollary 3.3, so that φ yields a power of i . The coefficient of $\mathbf{e}\left[\frac{ab}{2}(h, h)\right]$ appearing in the separation into cases in [Sh] is also clearly a power of i . However, the coefficient $\zeta_8^{-(p-q)\text{sgn}(cd)}$ is an odd power of ζ_8 , so that our expressions do not coincide with those of [Sh]. The reason is probably the fact that in the first part of the proposition of [Sh] in question the number $c(h, k)_\sigma$ is asserted to be the coefficient in a matrix representing $\rho_M(\sigma)$ but the formula given there corresponds to the tensor product of the operators $\mathbf{r}_0(\sigma)$ over all primes.

We now compare to the two statements about ρ_M -images of elements asserted in [BS]. The first is Lemma 2.1 there, about the element R_d . Evaluating this element yields $\left(\begin{pmatrix} a & ad-1 \\ 1-ad & d(2-ad) \end{pmatrix}, (-1)^{\sigma(a)\sigma(d)} \sqrt{j(R_d, \tau)}\right)$,

where we denote R_d also the image of this element in $SL_2(\mathbb{Z})$ (and the square root is the usual one). First we claim that the elements appearing with power m are 1 also for odd m . Indeed, a is odd, and the quadratic reciprocity law reduces this expression to $(\frac{c}{a})$. But $c = 1 - ad \equiv 1 \pmod{a}$, which proves this assertion. The remaining expressions (regardless of the parity of m) are easily seen to be $\prod_{p|N} \frac{\gamma(af_p)}{\gamma(f_p)}$ by Lemma 4.2 (since a is coprime to N). The usual argument allows to replace a by d here, and Milgram's identity for our D_M and for D_M with the bilinear form multiplied by d (this is again non-degenerate since d is coprime to Δ_M by Lemma 2.1) shows that this is indeed the quotient of the Gauss sums asserted in [BS]. The second element whose action is asserted in [BS] is $U^m = ST^m S^{-1}$ presented in Lemma 2.3 there. The substitution $\nu = \beta + \lambda$ in the expression for $\rho_M(U^m)$ from [BS] yields (the complex conjugate of) a Gauss sum of the form appearing in Proposition 3.8 and Theorem 3.9 of [Sche], and we note that the b -entry of U^m is zero. Since Lemma 4.2 shows that the complex conjugate of the expression denoted ε_c in [Sche] is precisely the product of the coefficients ξ_p from Theorem 8.1, our assertion about $\rho_M(U^m)$ is consistent with that of [BS]. In fact, since the assertion of [BS] is obtained by a direct evaluation using S and T^m , this suggests an alternative proof to Theorem 3.9 of [Sche].

The fact that a lower triangular unimodular matrix with c -entry positive and divisible by N (with $\varepsilon = 1$) lies in $\ker \rho_M$ (either by Proposition 8.2 or simply because it is the conjugate of T^{-c} by S) is related to the generalization appearing in [MH] of Braun's formula, namely

$$\sum_{\eta \in M/cM} \mathbf{e}\left(\frac{\eta^2}{2c}\right) = \zeta_8^{\text{sgn}(M)} c^{n/2} \sqrt{|M^*/M|}$$

for any c which is divisible by N . In fact, [MH] assumes that $2\Delta_M|c$, but the proof requires only that multiplying the bilinear form on M^* by c transforms it into an even lattice. It follows that the condition $N|c$ (which is weaker since $N|2\Delta_M$) suffices. Indeed, the fact that multiplying the tensor product of the operators $\mathbf{r}_0(A)$ over all the primes by $\prod_p \overline{\gamma(cf_p)} = \zeta_8^{-\text{sgn}(M)}$ (since $c > 0$) cancels the coefficients obtained in Theorem 4.3 immediately proves this generalization of Braun's formula. For $c < 0$ we obtain the complex conjugate of the number on the right hand side of Braun's formula (when written with $|c|^{n/2}$), since then $\prod_p \overline{\gamma(cf_p)}$ gives $\zeta_8^{+\text{sgn}(M)}$ (or since multiplying the bilinear form by c inverts the sign of the signature).

9 Odd Lattices and Further Generalizations

In this Section we consider the changes we have to introduce if we take the lattice M to be odd rather than even, and describe briefly further possible generalizations.

9.1 The Group Corresponding to Odd Lattices

The formulae in [Sh], and also the (generalized) formula of Braun, do not assume that the lattice is even. Hence one may ask what results can one obtain with odd lattices. To answer this question we need to consider only a part of the elements in $SL_2(\mathbb{Z})$ or in $Mp_2(\mathbb{Z})$, as described in the following

Lemma 9.1. *For $A \in SL_2(\mathbb{Z})$, the following conditions are equivalent: (i) ac and bd are even. (ii) ab and cd are even. (iii) A lies in $\Gamma(2) \cup ST(2)$.*

Proof. The SL_2 condition implies that either ad or bc are odd, and both conditions (i) and (ii) are equivalent to the other pair being even. Condition (iii) is also equivalent, since in $\Gamma(2)$ we have odd a and d and even b and c , and in $ST(2) = \Gamma(2)S$ we have even a and d and odd b and c . This proves the lemma. \square

We denote the set of elements characterized by Lemma 9.1 by Γ_{odd} . It is then clear that Γ_{odd} is a subgroup of $SL_2(\mathbb{Z})$, which has index 3 there (since the index of $\Gamma(2)$ is 6 and $S^2 \in \Gamma(2)$). Clearly T^2 and S are in Γ_{odd} , and in fact these elements generate this group. Indeed, for a matrix with $|d| < |c|$ multiplication by S from the right interchanges c and d up to sign, and if $|d| > |c|$ then multiplying from the right by some even power of T gives a matrix with $|d| < |c|$. Then an induction process on $|c|$ shows that any element of Γ_{odd} is generated by T^2 and S . The same argument holds in $Mp_2(\mathbb{Z})$, since $Z^2 = S^4$ is in the $Mp_2(\mathbb{Z})$ -version of Γ_{odd} . Hence the “metaplectic” Γ_{odd} is a double cover of the “usual” Γ_{odd} . We can define in a similar manner $\Gamma_{odd}(\mathbb{Z}_p)$ for any prime p , which is the entire group $SL_2(\mathbb{Z}_p)$ for odd p and is a subgroup of index 3 in $SL_2(\mathbb{Z}_p)$ if $p = 2$. It is clear that an element $A \in SL_2(\mathbb{Z})$ lies in Γ_{odd} if and only if it lies in $\Gamma_{odd}(\mathbb{Z}_2)$.

Now, if M is an odd lattice then D_M is no longer a discriminant form (since $\frac{\gamma^2}{2}$ is not well-defined in \mathbb{Q}/\mathbb{Z} there). Hence the construction from Section 1 does not work in this case. However, We show below that in this case the process of Sections 7 and 8 yields a representation of Γ_{odd} . As usual, it yields a representation of the “metaplectic” Γ_{odd} for odd m and of the “usual” one for even m . For simplicity and generality, we shall refer from now on only to the “metaplectic” one. Having said so, it is not surprising that the construction of Section 1 does not work here, since $T \notin \Gamma_{odd}$.

We first observe that the decomposition $V_{\rho_M} = \bigoplus_p V_{\rho_{M_p}}$ does not depend on the lattice being even, hence holds also here. It follows that by proving Lemma 2.2 with $\Gamma_{odd}(\mathbb{Z}_p)$, one obtains a representation of (the “metaplectic”) Γ_{odd} , which we continue to denote ρ_{M_p} , on the corresponding space $V_{\rho_{M_p}}$. All that remains is to follow the proofs of all the assertions along the way, and verify that they hold under the following change in assumptions: Instead of the condition that the lattice is even, we consider only $A \in \Gamma_{odd}$. Hence we allow the (previously excluded) Jordan component $1_t^{\varepsilon^n}$ in M_2 , but assume that the conditions from Lemma 9.1 hold. Clearly only the parts concerning $p = 2$ require attention. We note that Lemma 2.1 no longer holds for $p = 2$: Indeed,

the level of any odd lattice (which is defined in the same manner) must be even, but if M_2 is unimodular (and odd) then 2 does not divide Δ_M . Apart from this difference, the assertions about \tilde{N} , N , and Δ_M in Section 8 continue to hold. We note that Theorem 3.2 no longer holds in general, since in Lemma 3.1 we use conjugation by $T \notin \Gamma_{\text{odd}}$. Indeed, Theorem 9.2 below presents some cases in which $\ker \rho_M$ lies over a proper subgroup of $\Gamma(N)$. Moreover, Corollary 3.3 does not hold for an odd lattice, as the level of any odd lattice such that M_2 is unimodular is divisible only by 2.

Our key argument in Section 7 treats more general rings, and in general more than a simple partition into “even” and “odd” lattices may exist. Consider a lattice M over a ring R in which 2 is nonzero and not invertible. If we let I be the ideal generated by all the elements x^2 with $x \in M$ and $J = (2R : I)$ be the ideal containing all those elements $t \in R$ such that $2|ts$ for any $s \in I$, then Lemma 9.1 remains true with “even” replaced by “in J ”, “odd” replaced by “prime to J ”, and $\Gamma(2)$ replaced by the group $\Gamma_0^0(R, J)$ consisting of matrices in $SL_2(R)$ with b and c entries lying in J . Since for $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$ and M not even we have $J = 2R$, the fact that $\Gamma_0^0(2) = \Gamma(2)$ over these two rings implies this is indeed the correct generalization. A special case which is of interest to us is where $R = \mathcal{O}$ is the ring of integers in an extension of \mathbb{Q}_2 with ramification index $e = v(2)$. In this case, for any $0 < t < e$ there exist “non-even” lattices for which $I = \pi^t \mathcal{O}$ and then $J = \pi^{e-t} \mathcal{O}$.

9.2 Results for Odd Lattices

Let us now assume that we work with matrices in the corresponding Γ_{odd} at every stage, and verify that the proof of every assertion holds. Consider the lattice M over the ring \mathcal{O} from the end of the previous paragraph (with e and t defined there) and $A \in \Gamma_0^0(\mathcal{O}, \pi^{e-t} \mathcal{O}) \cup ST_0^0(\mathcal{O}, \pi^{e-t} \mathcal{O})$ (i.e., ab and cd , or equivalently ac and bd , are divisible by π^{e-t} by the extension of Lemma 9.1). We see in the proof of Proposition 7.1 that for $c = 0$, even though $\frac{x^2}{2}$ is no longer congruent to $d^2 \frac{\gamma^2}{2}$ modulo \mathcal{O} , they become congruent after multiplying by ab (part (ii) of Lemma 9.1). The same part shows that also for $c \neq 0$ all the elements which are asserted to be in \mathcal{O} indeed lie in this ring. The Gauss sum remains well-defined, by an expansion similar to that of the previous paragraph. Corollary 7.2 remains unchanged, and so are the assertions regarding the subgroups $D_{M,c}$ and D_M^c . Thus, if $\pi^{e-t}|c$, D_M^{c*} is still a well-defined coset of D_M^c and Lemma 7.3 (hence also Corollary 7.4) continues to hold. Moreover, in this case as well as in the even lattice case we see that any element in $M_{\mathbb{K}}$ which satisfies the D_M^{c*} condition (now for elements in the inverse image of $D_{M,c}$ inside $M^* \subseteq M_{\mathbb{K}}$) must lie in M^* (since $M \subseteq D_{M,c}$ and $c \frac{\mu^2}{2} \in \mathcal{O}$ for $\mu \in M$). Since translation by an element of D_M^c (and in particular an element of M) does not affect the D_M^{c*} condition, we deduce that modulo M we obtain the same D_M^{c*} . On the other hand, for odd c (with trivial $D_{M,c}$ in the local case) the expression $c \frac{\mu^2}{2} + (\beta, \mu) \in \mathbb{F}/\mathcal{O}$ depends on the choice of μ in M (or more generally in the inverse image of $D_{M,c}$ in $M_{\mathbb{K}}$). It follows that as a subset of

$M_{\mathbb{K}}$, D_M^{c*} is still a coset of D_M^c , but it is no longer contained in M^* and does not have a quotient in D_M . However, for $R = \mathbb{Z}_2$ (or $R = \mathbb{Z}$) Lemma 7.5 shows that the element x_c still lies in D_M^{c*} in this sense. Corollary 7.6 continues to hold as well, but with a rather than a_2 : Recall that now we can have $t_c \neq 0$ also for odd c (hence even a by part (i) of Lemma 9.1). However, since the assertion of Lemma 7.3 is void for $c \in \mathcal{O}^*$, its statement extends trivially to this case. Since $\frac{\beta^2}{2} = \frac{x_c^2}{2} + c\frac{\beta_c^2}{2}$, we can extend this for odd (hence invertible) c to $\beta \in M^*$ and define $\frac{\beta_c^2}{2}$ as $\frac{1}{c}(\frac{\beta^2}{2} - \frac{x_c^2}{2})$. This implies that Corollary 7.4 extends also to this case (note however the remark about elements with odd c in $\ker \rho_M$ below).

Now, define the Weil index of the odd Jordan component $1_t^{\varepsilon n}$ to be ζ_8^t , as the natural extension of Proposition 4.1. Indeed, since the Weil index is determined by the p -adic vector space, we know that $1_t^{\varepsilon n}$ should have the same Weil index as any $q_t^{\varepsilon n}$ with q an even power of 2. However, Note that the Gauss sum of Proposition 4.1 is no longer well-defined in this case. The same statement holds for Milgram's formula, but the Weil Reciprocity Law is not affected. Lemma 4.2 extends trivially to odd lattices, and so does Theorem 4.3 for odd p or even c (note that $\frac{(\eta+c\lambda)^2}{2c} = \frac{\eta^2}{2c} + (\eta, \lambda) + \frac{c\lambda^2}{2}$). For $p = 2$ and odd c the sum is over one element, but we have to see that it is well-defined (hence equals 1). Indeed, a is even according to condition (i) of Lemma 9.1, so that any choice of $x_c \in D_M$ will do. This also shows that with the actual choice of $x_c \in M_{\mathbb{Q}_2}$ (which is not in M^*) the sum is also well-defined. Thus, the assertions in Section 8 continue to hold if we assume $A \in \Gamma_{\text{odd}}$. In particular this proves the assertion in the first part of Proposition 1.6 of [Sh] in full generality—note that Condition (1.21) in this reference is equivalent to the statement that $A \in \Gamma_{\text{odd}}$ if M is not even, as part (ii) of Lemma 9.1 shows. Thus, the proof of Theorem 8.1 extends to give the same formula for $\rho_M(A)$ for $A \in \Gamma_{\text{odd}}$ if M is odd, except for an additional factor $\zeta_8^{(a-a_2)c_2t_c}$ which accounts for the fact that for odd c the factor $\chi_2(\frac{a}{c}\frac{x_c^2}{2})$ does not cancel the corresponding Weil index.

As for Proposition 8.2, we only have to evaluate the effect of the odd Jordan component $1_t^{\varepsilon n}$. This component does not affect Δ_{M_2} and has the Weil index ζ_8^t , and it yields an additional case in which \tilde{N} is odd and $2||N$. The results are as follows. For the assertion that $\ker \rho_M$ contains $\Gamma(N)$ (or more precisely, contains a double cover of that group if m is even and a lift of it if m is odd) we recall that N is always even, hence a is odd and the factor $\zeta_8^{(a-a_2)c_2t_c}$ equals 1. Since the odd Jordan component $1_t^{\varepsilon n}$ only contributes $\zeta_8^{(a-1)t}$ to φ (and $Nt|(a-1)t$ on $\Gamma(N)$), it follows that $\ker \rho_M$ still lies over a group containing $\Gamma(N)$ if $8|Nt$ or if m is odd and $4|Nt$ (possibly with a change of the lift). Since in any case $8|2Nt$ for even m and $4|2Nt$ for odd m (as $2|N$ and if $4 \nmid N$ then $M_2 \cong 1_t^{\varepsilon n} 2_{II}^{\delta(m-n)}$ hence t , n , and m have the same parity), it remains to consider the cases $Nt \equiv 4 \pmod{8}$ for even m and $Nt \equiv 2 \pmod{4}$ for odd m . Now, since N is even, we know that for an element in $\Gamma(N)$ we have either $a \equiv d \equiv 1 \pmod{2N}$ or $a \equiv d \equiv 1+N \pmod{2N}$, and the elements satisfying the first congruence form a subgroup $\tilde{\Gamma}$ of index 2 in $\Gamma(N)$. The previous argument shows that under the last assumption on Nt the character φ attains $(\pm 1)^m$ on

$\tilde{\Gamma}$ but not on the other elements of $\Gamma(N)$. Hence an element of $\ker \rho_M$ lying over $\Gamma(N)$ must lie over $\tilde{\Gamma}$ in this case. Therefore, $\tilde{\Gamma}$ is normal in Γ_{odd} for any such N (as can be verified for any N by conjugation), even though not in $SL_2(\mathbb{Z})$ itself. Summarizing, $\ker \rho_M$ always lies over a subgroup of Γ which contains $\tilde{\Gamma}$. We know precisely when this subgroup contains $\Gamma(N)$, and it remains to see, if $N = 2\tilde{N}$, which further elements of $\Gamma \setminus \Gamma(N)$ act trivially via ρ_M .

Now, if $4|\tilde{N}$ then the same argument we had for the even lattice case shows that $\ker \rho_M$ lies over Γ unless $4|\tilde{N}$, m is even and $v_2(\Delta_M)$ is odd. If $2|\tilde{N}$ (and $N = 2\tilde{N}$) then the odd lattice M_2 must be of the form $1_t^{\varepsilon n} 2_s^{\delta, m-n}$. Now, for odd m no element outside $\Gamma(N)$ can be in $\ker \rho_M$ (since for $a \equiv 2 \pmod{4}$ we have $\gamma(f_2)^{a-1} = \pm i$). Hence assume m is even. Then the character φ depends on the value of a and d modulo 8 (recall that since $4|N|b, c$ we have $ad \equiv 1 \pmod{16}$ and hence $a \equiv d \pmod{8}$). Explicitly, t , n , and s have the same parity, and we have seen above that $\ker \rho_M$ lies over a subgroup containing $\Gamma(N)$ if and only if this parity is even (for $8|Nt$ to hold). In this case $\ker \rho_M$ lies over Γ if and only if $\gamma(f_2)^2 = 1$ (as in the even lattice case). On the other hand, if this parity is odd then the fact that φ attains ± 1 on elements of Γ implies that $\ker \rho_M$ lies over a subgroup of index 2 in Γ which is distinct from $\Gamma(N)$ but contains $\tilde{\Gamma}$. The elements of $\ker \rho_M$ which do not lie over $\tilde{\Gamma}$ are characterized by the congruence $a \equiv d \equiv 1 - (-1)^{\varepsilon(\tilde{N}/2)} \tilde{N} \pmod{2N}$ (i.e., having residue -1 modulo 8) if $\gamma(f_2)^2 = 1$ and by $a \equiv d \equiv 1 + (-1)^{\varepsilon(\tilde{N}/2)} \tilde{N} \pmod{2N}$ (i.e., residue -5 modulo 8) if $\gamma(f_2)^2 = -1$. To see this, recall that $(\frac{\Delta_{M_2}}{a}) = (\frac{2}{a})$ equals 1 for a being ± 1 modulo 8 and equals -1 for a being ± 5 modulo 8, and that $(-1)^{\varepsilon(\tilde{N}/2)} \tilde{N} \equiv 2 \pmod{8}$ wherever $2|\tilde{N}$.

It remains to consider the case where \tilde{N} is odd, so that $M_2 \cong 1_t^{\varepsilon m}$ is unimodular (i.e., $\Delta_{M_2} = 1$) and $2|\tilde{N}$. In this case $t \equiv m \pmod{2}$ implies that 4 cannot divide Nt for odd m , and since for even m we need $8|Nt$ we find that $\Gamma(N) \subseteq \ker \rho_M$ only for even m and with $\gamma(f_2)^2 = 1$. Note that since $N = 2\tilde{N}$ we have $\Gamma \cap \Gamma_{odd} = \Gamma(N)$ (as $\Gamma \subseteq \Gamma_0^1(N)$). However, since here x_c does not belong to M^* , the proof of Theorem 8.1 yields a similar expression for $\rho_M(A)$ but with summation on $\beta \in D_M^c$ (rather than D_M^{c*}) and with $x_c = 0$. This presentation has the advantage that it does not contain $\frac{\beta^2}{2}$, which is non-zero for $\beta = 0$ and odd c by our choice of x_c . Then the argument preceding Proposition 8.2 implies that $\ker \rho_M$ lies over a subgroup of $\Gamma(\tilde{N}) \cap \Gamma_{odd}$ (rather than Γ). Hence we must check which elements in $(\Gamma(\tilde{N}) \cap \Gamma_{odd}) \setminus \Gamma(N)$, i.e., with even a and odd c , are in $\ker \rho_M$. Now, for odd p this presentation yields the same formulae, and for $p = 2$ we can write ξ_2 simply as $\varepsilon^m (\frac{a}{c_2})^m \gamma(cf_2)$. As for elements in $\Gamma(\tilde{N})$ we have $\xi_p = 1$ for any odd p , we obtain that the total value of φ on the elements in question is $\varepsilon^m (\frac{a}{c_2})^m \zeta_8^{-c_2 t}$ (with $c_2 = c$ odd). This shows that $\ker \rho_M$ can contain elements outside $\Gamma(N)$ only if m is even and $\gamma(f_2) = 1$, in which case it is a double cover of $\Gamma(\tilde{N}) \cap \Gamma_{odd}$. These considerations also show that the part (i) \Rightarrow (iii) in Lemma 2.1 does not hold for odd lattices and $p = 2$, as ρ_{M_2} may be non-trivial also when M_2 is unimodular. In fact, this situation

occurs whenever $M_2 \cong 1_t^{\varepsilon m}$ with non-zero t , and in particular when m is odd.

To complete the picture, the (generalized) formula of Braun is obtained, by the same argument, also for odd lattices. Indeed, [MH] makes no assumption on the parity of the lattice under consideration when proving this formula.

As in the case for the even lattices, one can define theta functions also for odd lattices. They can be defined in the full generality of Section 4 of [B1]: For an element $v \in G(M)$, a polynomial p on $M_{\mathbb{R}}$ which is homogenous of degree (m_+, m_-) with respect to v , and elements α and β in $M_{\mathbb{R}}$, one defines $\Theta_M(\tau, v; \binom{\alpha}{\beta}, p) = \sum_{\gamma \in D_M} \theta_{M+\gamma}(\tau, v; \binom{\alpha}{\beta}, p) e_{\gamma}$ with

$$\theta_{M+\gamma}\left(\tau, v; \binom{\alpha}{\beta}, p\right) = \sum_{\lambda \in M+\gamma} e^{-\frac{\Delta_v}{8\pi y}(p)(\lambda+\beta)} e\left[\tau \frac{(\lambda+\beta)_{v+}^2}{2} + \bar{\tau} \frac{(\lambda+\beta)_{v-}^2}{2} - \left(\lambda + \frac{\beta}{2}, \alpha\right)\right]$$

(using the notation of [Z2]). The results for odd lattices are summarized in the following

Theorem 9.2. *For an odd lattice M we have a Weil representation ρ_M of (the “metaplectic”) Γ_{odd} on the space $\mathbb{C}[D_M]$, and $\rho_M(A, \varepsilon \sqrt{j(A, \tau)})$ for $A \in \Gamma_{\text{odd}}$ is described by the formula in Theorem 8.1 with ξ_2 multiplied by $\zeta_8^{(a-a_2)c_2 t_1}$. The kernel of ρ_M lies over the group Γ described before Proposition 8.2 (and it is contained in Γ_{odd}), except for the following cases: (i) $4||\tilde{N}$, $N = 2\tilde{N}$, m is even, and $v_2(\Delta_M)$ is odd; (ii) $2||\tilde{N}$, $N = 2\tilde{N}$, m is odd; (iii) $2||\tilde{N}$, $N = 2\tilde{N}$, m and t_1 are even, $\gamma(f_2)^2 \neq 1$; (iv) \tilde{N} is odd, $\gamma(f_2) = -1$ (hence m is even); (v) $4||\tilde{N} = N$, m is even, t_1 is odd; (vi) $2||\tilde{N} = N$, $\gamma(f_2)^2 \neq 1$; (vii) \tilde{N} is odd, $\gamma(f_2)^2 \neq 1$; (viii) $2||\tilde{N}$, $N = 2\tilde{N}$, m is even, t_1 is odd; (ix) \tilde{N} is odd, $\gamma(f_2) = 1$ (hence m is even). In cases (i) – (iv) kernel of ρ_M lies over $\Gamma(N)$ (and $\Gamma(N) \subsetneq \Gamma$). In cases (v) – (vii) it lies over $\tilde{\Gamma}$. In case (viii) it lies over a proper subgroup of Γ , which properly contains $\tilde{\Gamma}$ and does not equal $\Gamma(N)$ (there are two such groups, and the explicit one depends on whether $\gamma(f_2)^2$ equals 1 or -1). Finally, in case (ix) it lies in $\Gamma(\tilde{N}) \cap \Gamma_{\text{odd}}$ (which is not contained in Γ). In any case it is a double cover of that group if m is even and it is a lift of it if m is odd. The theta function of M is modular of weight $(m_+ + \frac{b_+}{2}, m_- + \frac{b_-}{2})$ and representation ρ_M with respect to Γ_{odd} , in the sense that*

$$\Theta_M\left(A\tau, v; A\binom{\alpha}{\beta}, p\right) = j(A, \tau)^{m_+ + b_+/2} \overline{j(A, \tau)^{m_- + b_-/2}} \rho_M(A) \Theta_M\left(\tau, v; \binom{\alpha}{\beta}, p\right),$$

where $\Gamma_{\text{odd}} \subseteq SL_2(\mathbb{R})$ acts on $M_{\mathbb{R}}^2 = M \otimes_{\mathbb{R}} \mathbb{R}^2$ through the natural action on \mathbb{R}^2 .

Proof. All the assertions except for the one about the theta function have been proven above. The proof regarding the theta function follows that of Theorem 4.1 of [B1], by checking the action of the generators T^2 and S of Γ_{odd} (note that the action on α and β can indeed be described as above, by letting subgroups of $SL_2(\mathbb{R})$ act on $M_{\mathbb{R}}^2$ via the action on the component \mathbb{R}^2). Indeed, the argument used for the evaluation for T [B1] holds here for T^2 , and the evaluation for S makes no use of the hypothesis that M is even. \square

9.3 Weil Representations of Larger Global Groups

The generality of the results of Section 7 suggests that it may be possible to generalize Theorem 8.1 (and also the first part of Theorem 9.2) from \mathbb{Z} -lattices to \mathcal{O} -lattices where \mathcal{O} is now the ring of integers in a number field \mathbb{F} other than \mathbb{Q} . To carry on this task, several difficulties have to be overcome. The first one is that in general the appropriate group $Mp_2(\mathcal{O})$ is no longer generated by two elements with two simple relations, so that the process of Section 1, which is strongly based on the structure of the specific group $Mp_2(\mathbb{Z})$, can no longer be used. This, however, is not a serious problem, since the arguments in Sections 7, 8, and 9 do not use the process of Section 1.

A more serious problem may arise due to the following consideration. The local characters, which for $\mathcal{O} = \mathbb{Z}$ were the χ_p , must have two properties. First, we require the *product formula*, i.e., that $\prod_v \chi_v(x) = 1$ for all $x \in \mathbb{F}$, where the product is taken over all places of \mathbb{F} , finite or infinite. For $\mathbb{F} = \mathbb{Q}$ this was the formula $\mathbf{e}(x) = \prod_p \chi_p(x)$, since $\chi_\infty(x) = \mathbf{e}(-x)$ in this convention. This property is necessary for the Weil Reciprocity Law and other Adélic assertions in [W] and [Ge] to hold. The second property is that for any non-archimedean completion \mathbb{F}_v of \mathbb{F} we need that the character $\lambda = \chi_v$ on $(\mathbb{F}_v, +)$ will satisfy the condition that $\lambda(x\mathcal{O}_v) = 1$ if and only if $x \in \mathcal{O}_v$. This is used both for the composition of λ with the quadratic (or bilinear) form on D_{M_v} to identify this group with its Pontryagin dual, and for the proof of Proposition 7.1. The natural canonical choice of composing χ_p with the trace from \mathbb{F}_v to \mathbb{Q}_p (for the appropriate p) has the first property, but fails the second wherever v is ramified. Since in any number field other than \mathbb{Q} there exist ramified primes, there is no field (other than \mathbb{Q}) for which our proof can extend with this choice of characters.

Another fact which we have implicitly used and holds for \mathbb{Q} but not for other number fields, is that we have only one infinite place and only one place over 2. Indeed, we have treated our “global” group $Mp_2(\mathbb{Z})$ as embedded in $Mp_2(\mathbb{R})$, for the odd primes we have used the representation of $SL_2(\mathbb{Z}_p)$ from Corollary 6.4, and in Corollary 6.7 we have made all the adjustment in the remaining place $p = 2$. This is clearly not the best choice if there are more infinite places and more places over 2. We note that complex infinite places can probably be “ignored”, since the metaplectic cover splits over \mathbb{C} . Therefore this approach cannot work for totally complex fields. It is very probable that if the rank of the lattice is odd then the representation is of a non-trivial double cover $Mp_2(\mathcal{O})$ of $SL_2(\mathcal{O})$. This is indeed the case if the rank of \mathbb{F} over \mathbb{Q} is odd: In this case the lattice has odd rank over \mathbb{Z} , so that the restriction of this representation to the inverse image of $SL_2(\mathbb{Z})$ becomes the non-trivial double cover $Mp_2(\mathbb{Z})$. If \mathbb{F} is totally complex then by Theorem 6.3 the only places where we can have a non-trivial double cover are those lying over 2. Hence expects that $SL_2(\mathcal{O}_v)$ cannot be lifted to $Mp_2(\mathbb{F}_v)$ for these places even if the degree of \mathbb{F}_v over \mathbb{Q}_2 is even. We leave the more detailed analysis for future work.

Finally we remark that our results extend to the case of a function field (with characteristic $\neq 2$). In this case every lattice is even, and the process of

Section 1 gives a representation of $SL_2(\mathbb{Z})$, and in fact of $SL_2(\mathbb{F}_p)$ (where p is the characteristic of the function field). Moreover, the results of Sections 5, 6, and 7 hold here as well, so that one can use the tools of Section 8 in order to obtain the explicit formulae for a representation of SL_2 over a ring which is finite over $\mathbb{F}_p[X]$. This representation is the Weil representation corresponding to a lattice over that ring.

Another generalization of our results may be carried out as follows. For a \mathbb{Z} -lattice M , consider the ring $End(M)$, and consider the symplectic \mathbb{Z} -module $M \times M$ with the anti-symmetrization of the bilinear form. Now consider the group $Sp(M \times M)$, as the group of 2×2 matrices over $End(M)$ satisfying the symplectic condition. It contains $SL_2(\mathbb{Z})$ as the subgroup in which the coordinates lie in the image of \mathbb{Z} in $End(M)$. We consider $Sp(M \times M)$ as a subgroup of $Sp(M_{\mathbb{Q}_p} \times M_{\mathbb{Q}_p})$, whose closure is $Sp(M_p \times M_p)$ for every prime p . It should not be hard to show that the action of the S^1 -cover of $Sp(M_p \times M_p)$ preserves the finite-dimensional subspace of $\mathcal{S}(M_{\mathbb{Q}_p})$ defined above Lemma 2.2, and that for primes p satisfying the conditions of Lemma 2.1 the thus obtained representation is trivial. Then it should be possible to combine the methods of Section 5 and the ideas of [Ra] in order to obtain the metaplectic double covers, and hopefully arguments as those of Section 6 will give the splitting of this double cover over local fields of odd residue characteristic. it is reasonable to conjecture that a similar tensor product argument yields a representation of a double cover of $Sp(M \times M)$, whose restriction to $Mp_2(\mathbb{Z})$ is our ρ_M . Applying then the methods of Sections 7 and 8 one may obtain the general formulae for this representation. For an \mathcal{O} -lattice M with \mathcal{O} the ring of integers in a number field which is larger than \mathbb{Q} , considering \mathcal{O} as a subring of $End(M)$ (as a \mathbb{Z} -lattice) might yield the formulae for the Weil representation of $Mp_2(\mathcal{O})$ as a subgroup of the double cover of $Sp(M \times M)$. All this, however, is the suggested subject for future work.

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